A Time-Varying Call Center Design via Lagrangian Mechanics

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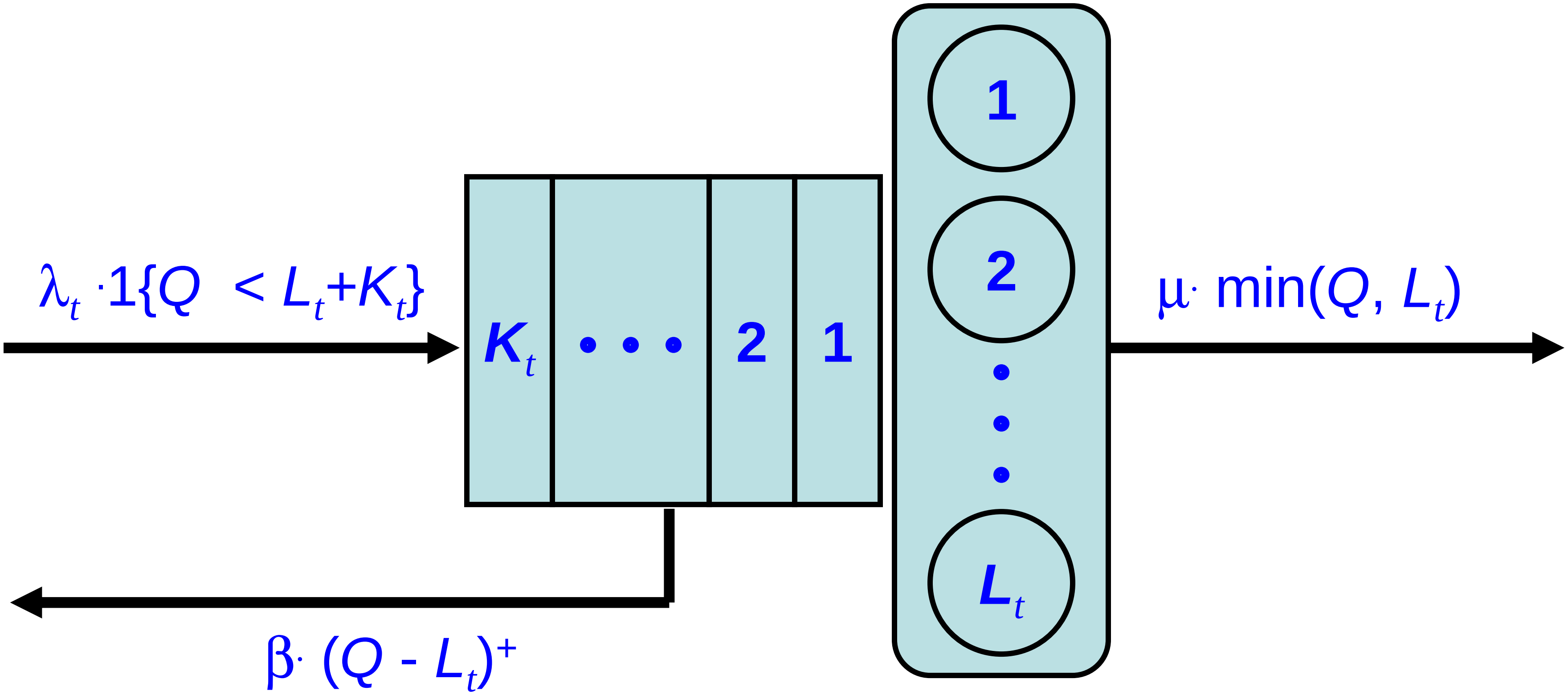
Basic Dynamic Call Center Model: The $M_t/M/L_t/K_t$ Queue With Abandonment



We have a non-homogeneous Poisson arrival rate λ_t, mean service time $1/\mu$, mean time to abandonment $1/\beta$, L call center agents, and $L + K$ telephone lines.[3]

Cost Structure and Service Level Targets

r = revenue per customer service completion,

$c(\cdot)$ = cost rate for the number of agents used

$d(\cdot)$ = cost rate for the number of telephone lines used.

ε_b = service level target for the percentage of customers blocked

ε_m = service level target for the percentage of abandoning customers.

Goal: Call Center Staffing For Profit Optimality

Find schedules K_t and L_t so that we can maximize the mean profit:

$$\int_0^T r\mu \cdot E\left[Q_{L/K}(t) \wedge L_t\right] - c\left(L_t\right) - d\left(K_t + L_t\right) dt$$

subject to the service level target constraints:

$$\int_0^T \beta \cdot E\left[\left(Q_{L/K}(t) - L_t\right)^+\right] dt \leq \varepsilon_a \cdot \left(Q_{L/K}(0) + \int_0^T \lambda_t \, dt \right)$$

and $\int_0^T \lambda_t \cdot P\left\{Q_{L/K}(t) = K_t + L_t\right\} dt \leq \varepsilon_b \cdot \left(Q_{L/K}(0) + \int_0^T \lambda_t \, dt \right)$

Simplifying Cost Structure Assumptions

To simplify our analysis, we assume that the cost functions are increasing and concave (economies of scale).

Competing Lagrangian Terms

$$\ell_1(p, q, \dot{q}) \equiv (p - \tau) \cdot \gamma \cdot q - c(0) - d(0)$$

$$\ell_2(p, q) \equiv (p - \sigma) \cdot \beta \cdot q - c(0) - d(q)$$

$$\ell_3(p, q) \equiv (p + r)\mu \cdot q - c(q) - d(q)$$

Steps 1 & 2: Solving the Euler- Lagrange Equations for Optimality

If $\ell_1(p,q)$ is dominant, then solve the differential equations

$\dot{q} = \lambda - \gamma q$ and $\dot{p} = (p - \tau)\gamma$ (busy mode).

If $\ell_2(p,q)$ is dominant, then solve the differential equations

$\dot{q} = \lambda - \beta q$ and $\dot{p} = (p - \sigma)\beta - d'(q)$ (music mode).

If $\ell_3(p,q)$ is dominant, then solve the differential equations

$\dot{q} = \lambda - \mu q$ and $\dot{p} = (p + r)\mu - c'(q) - d'(q)$ (agent mode).

Finally, we are given the *initial queueing load* $q(0) = Q_{L/K}(0)$ and the *terminal opportunity cost per customer* value $p(T) = 0$.

Steps 1 & 2: Calibrating the Penalty Costs

Select $\sigma =$ penalty per customer abandonment

and $\tau =$ penalty per blocked customer such that

$$\int_0^T \beta \cdot \left(\left(q(t) - L_t \right)^+ - \left(q(t) - K_t - L_t \right)^+ \right) dt = \varepsilon_a \cdot \left(Q_{L/K}(0) + \int_0^T \lambda_t \, dt \right)$$

and

$$\int_0^T \gamma \cdot \left(q(t) - K_t - L_t \right)^+ dt = \varepsilon_b \cdot \left(Q_{L/K}(0) + \int_0^T \lambda_t \, dt \right).$$

Step 3: Finding the Optimizing Square Root Safety Factor

Find χ such that

$$\frac{r\lambda}{\sqrt{q}} \cdot \frac{\phi(\chi)}{\Phi(\chi)} + (c+d)\left(q + \chi\sqrt{q}\right)$$

$$= \min_{a \geq 0} \frac{r\lambda}{\sqrt{q}} \cdot \frac{\phi(a)}{\Phi(a)} + (c+d)\left(q + a\sqrt{q}\right)$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(y)\, dy.$$

Asymptotic Properties of the Erlang Blocking Formula

According the Jagerman, we have

$$\lim_{z \to \infty} \sqrt{z} \cdot b(a,z) = \frac{\phi(a)}{\Phi(a)},$$

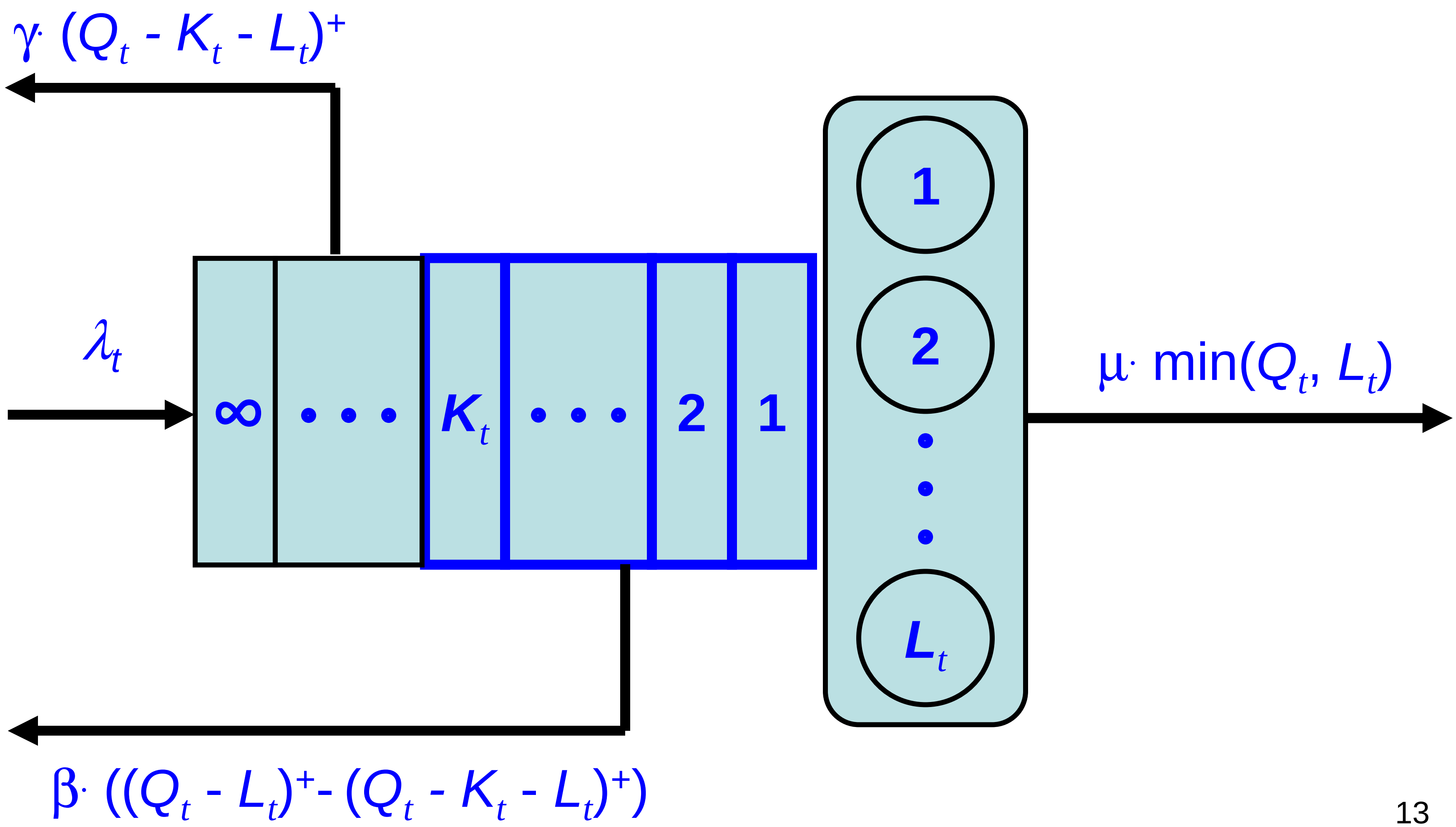
where

$$b(a,z) \equiv \frac{z^{\lceil z + a\sqrt{z} \rceil}}{\lceil z + a\sqrt{z} \rceil !} \Bigg/ \sum_{i=0}^{\lceil z + a\sqrt{z} \rceil} \frac{z^i}{i!}.$$

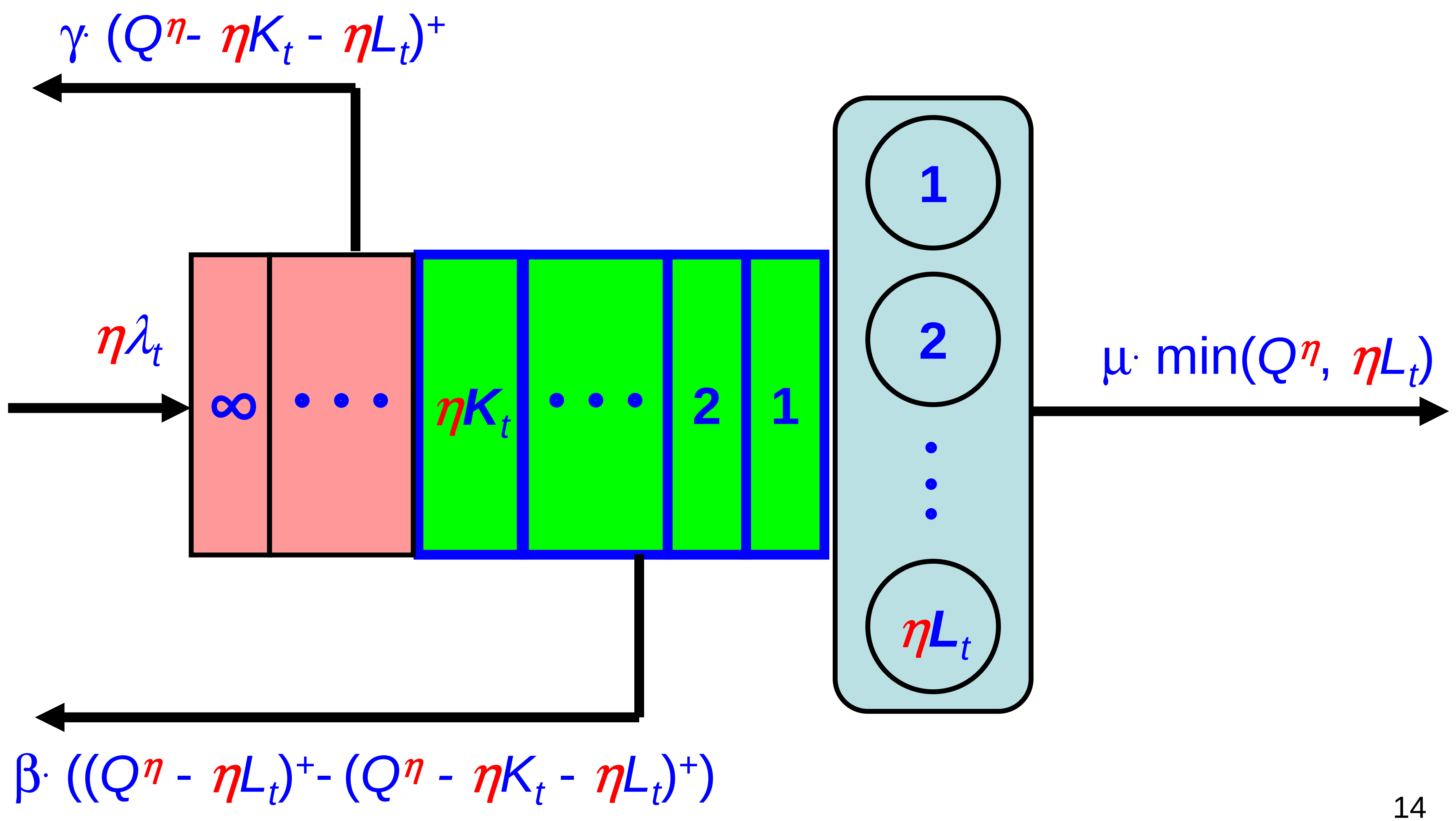
Step 4: The Fluid Modified Offered Load (FMOL) Schedule

Operational Mode	Dominant Lagrangian Term	Optimal K^*	Optimal L^*	Abandonment Rate	Blocking Rate
"busy signal"	$\ell_1(p, q)$	0	0	0	λ
"music"	$\ell_2(p, q)$	$\lceil q \rceil$	0	$\beta \cdot q \cdot (1 - b(0, q))$	$\lambda \cdot b(0, q)$
"agent"	$\ell_3(p, q)$	0	$\lceil q + \chi\sqrt{q} \rceil$	0	$\lambda \cdot b(\chi, q)$

Fast Abandonment Queueing Model to Approximate Call Center Model



Fast Abandonment Queueing Model
with Halfin-Whitt Scale Factor η



Fluid Model for the Call Center

Using results in Mandelbaum, Massey and Reiman and taking the limit as $\eta \to \infty$ gives us

$$\frac{1}{\eta}\mathbf{Q}^{\eta} \xrightarrow{a.s.} \mathbf{Q}^{(0)} = q \quad (u.o.c.)$$

Moreover, $\{\, q(t) \mid t \geq 0 \,\}$ is a dynamical system, where

$$\frac{d}{dt}q(t) = \lambda_t - \beta \cdot \left(\left(q(t) - L_t \right)^+ - \left(q(t) - K_t - L_t \right)^+ \right)$$

$$- \gamma \cdot \left(q(t) - K_t - L_t \right)^+ - \mu \cdot \left(q(t) \wedge L_t \right).$$

Fluid Approximation Of Call Center Staffing For Profit Optimality

Find a schedule K_t and L_t so that we can maximize the profit:

$$\int_0^T r\mu \cdot (q(t) \wedge L_t) - c(L_t) - d(K_t + L_t)\,dt$$ subject to the equality constraint:

$$\frac{d}{dt} q(t) = \lambda_t - \beta \cdot \left((q(t) - L_t)^+ - (q(t) - K_t - L_t)^+ \right)$$

$$-\gamma \cdot (q(t) - K_t - L_t)^+ - \mu \cdot (q(t) \wedge L_t).$$

with the integral QoS constraints:

$$\int_0^T \beta \cdot \left((q(t) - L_t)^+ - (q(t) - K_t - L_t)^+ \right) dt = \varepsilon_a \cdot \left(Q_{L/K}(0) + \int_0^T \lambda_t \, dt \right)$$

and $$\int_0^T \gamma \cdot (q(t) - K_t - L_t)^+ \, dt = \varepsilon_b \cdot \left(Q_{L/K}(0) + \int_0^T \lambda_t \, dt \right)$$

Equivalent Optimality Problem for Fluid Model Call Center

Find a schedule K_t and L_t so that we can maximize the profit:

$$\int_0^T r\mu\cdot\left(q(t)\wedge L_t\right)-\sigma\beta\cdot\left(\left(q(t)-L_t\right)^+-\left(q(t)-K_t-L_t\right)^+\right)$$

$$-\tau\gamma\cdot\left(q(t)-K_t-L_t\right)^+-c\left(L_t\right)-d\left(K_t+L_t\right)dt,$$

subject to the equality constraint:

$$\frac{d}{dt}q(t)=\lambda_t-\beta\cdot\left(\left(q(t)-L_t\right)^+-\left(q(t)-K_t-L_t\right)^+\right)$$

$$-\gamma\cdot\left(q(t)-K_t-L_t\right)^+-\mu\cdot\left(q(t)\wedge L_t\right).$$

Equivalent QoS Structure via Penalty Costs

Select $\sigma =$ penalty per customer abandonment

and $\tau =$ penalty per blocked customer such that

$$\int_0^T \beta \cdot \left(\left(q(t) - L_t \right)^+ - \left(q(t) - K_t - L_t \right)^+ \right) dt = \varepsilon_a \cdot \left(Q_{L/K}(0) + \int_0^T \lambda_t \, dt \right)$$

and

$$\int_0^T \gamma \cdot \left(q(t) - K_t - L_t \right)^+ dt = \varepsilon_b \cdot \left(Q_{L/K}(0) + \int_0^T \lambda_t \, dt \right).$$

Lagrangian Formulation for Penalty Cost Model

We maximize the time integral of the following Lagrangian,

$$\mathcal{L}\left(p, q, \dot{q}; K, L\right) = r\mu \cdot (q \wedge L) - \sigma\beta \cdot \left((q - L)^{+} - (q - K - L)^{+}\right)$$
$$-\tau\gamma \cdot (q - K - L)^{+} - c(L) - d(K + L)$$
$$+p \cdot \left\{\dot{q} - \lambda + \beta \cdot \left((q - L)^{+} - (q - K - L)^{+}\right)\right.$$
$$\left. + \gamma \cdot (q - K - L)^{+} + \mu \cdot (q \wedge L)\right\}.$$

Since the cost functions are increasing and concave, we have ...

Reduction to Competing Lagrangians

$$\max_{K,L\geq 0} \mathcal{L}\left(p,q,\dot{q};K,L\right)=\mathcal{L}\left(p,q,\dot{q};0,0\right)\vee\mathcal{L}\left(p,q,\dot{q};q,0\right)\vee\mathcal{L}\left(p,q,\dot{q};0,q\right)$$

where

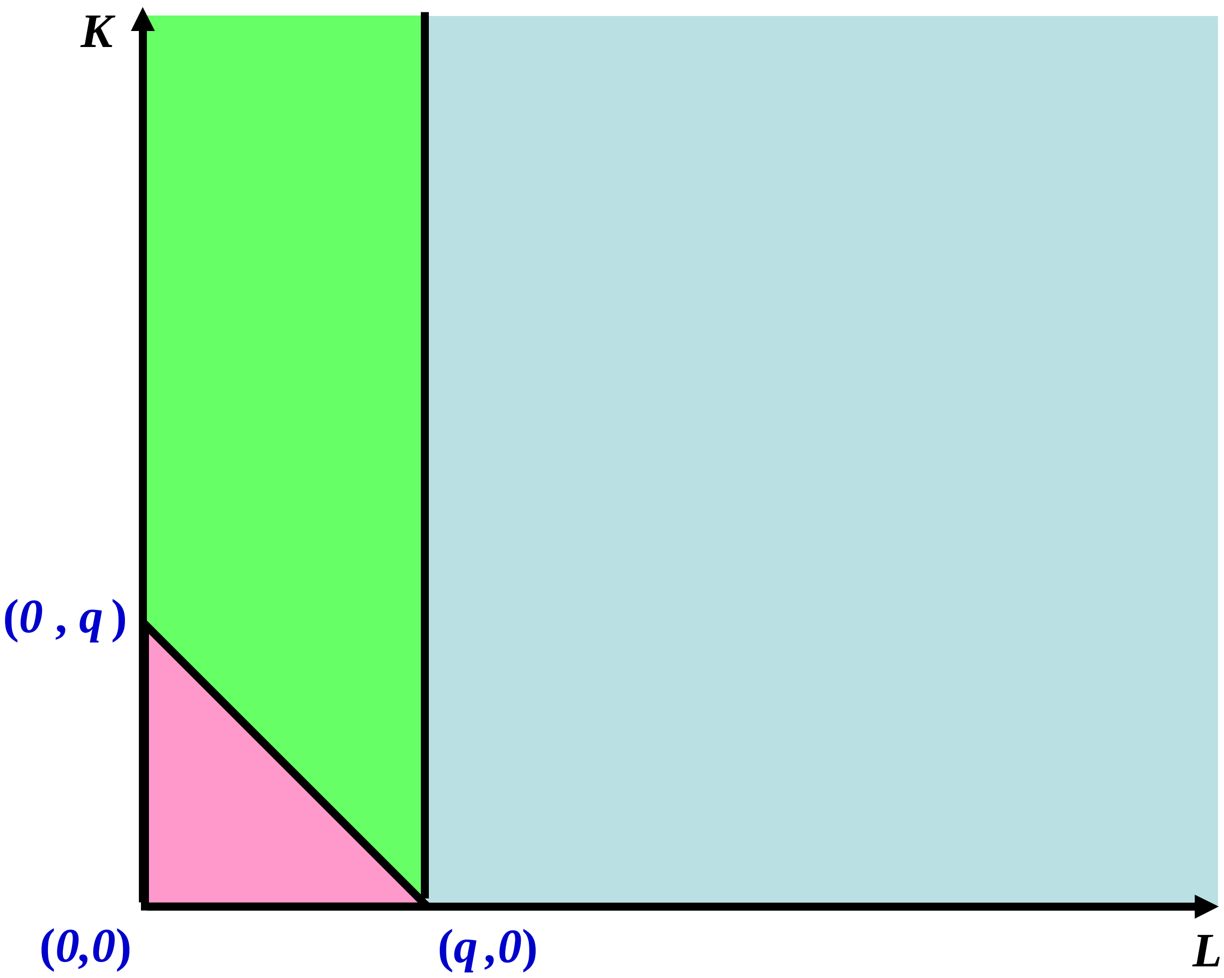
$$\mathcal{L}\left(p,q,\dot{q};0,0\right)=p\cdot\left(\dot{q}-\lambda\right)+(p-\tau)\gamma q,$$

$$\mathcal{L}\left(p,q,\dot{q};q,0\right)=p\cdot\left(\dot{q}-\lambda\right)+(p-\sigma)\beta q-d(q)$$

and

$$\mathcal{L}\left(p,q,\dot{q};0,q\right)=p\cdot\left(\dot{q}-\lambda\right)+(p+r)\mu q-c(q)-d(q).$$

Proof by Piecewise Linearity and Convexity



Hamilton Equations Follow from the Euler-Lagrange Equations for Optimality

If $\mathcal{L}$ is dominant, then the Euler-Lagrange equations are

$$\dot{q} = \lambda - \gamma q \text{ and } \dot{p} = (p - \tau)\gamma \quad \text{(busy mode).}$$

If $\mathcal{L}$ is dominant, then the Euler-Lagrange equations are

$$\dot{q} = \lambda - \beta q \text{ and } \dot{p} = (p - \sigma)\beta - d'(q) \quad \text{(music mode).}$$

If $\mathcal{L}$ is dominant then the Euler-Lagrange equations are

$$\dot{q} = \lambda - \mu q \text{ and } \dot{p} = (p + r)\mu - c'(q) - d'(q) \quad \text{(agent mode).}$$

Finally, we are given the *initial queueing load* $q(0)$ and the *terminal opportunity cost per customer* value $p(T) = 0$.

Optimal Staffing Schedule and Provisioning Plan

Optimal Agent Staffing Schedule (with telephone lines):

$$L_t^* = \begin{cases} q(t) & \text{if } \mathcal{L} \text{ is dominant,} \\ 0 & \text{otherwise } (\mathcal{L} \text{ or } \mathcal{L} \text{ dominant).} \end{cases}$$

Optimal Provisioning Plan (for telephone lines only):

$$L_t^* + K_t^* = \begin{cases} q(t) & \text{if } \mathcal{L} \text{ or } \mathcal{L} \text{ is dominant,} \\ 0 & \mathcal{L} \text{ is dominant.} \end{cases}$$

A Modified Offered Load Approximation for the Performance of a Dynamic Loss Model

Setting $q(t) = E[Q_\infty(t)]$, the mean of an $M_t/M/\infty$ queue, the *modified offered load approximation* (due to Jaegerman) for an $M_t/M/L/0$ queue is

$$\mathrm{P}\left\{Q_{L/0}(t) = L\right\} \approx \beta_L\left(q(t)\right) = \mathrm{P}\left\{Q_\infty(t) = L \,\middle|\, Q_\infty(t) \leq L\right\}.$$

where

$$\frac{d}{dt} q(t) = \lambda(t) - \mu \cdot q(t).$$

Non-Linear Cost Numerical Example

$$\lambda(t) = 300 + 300 \cdot \sin(\pi t / 12), \mu = 6, \beta = 12$$

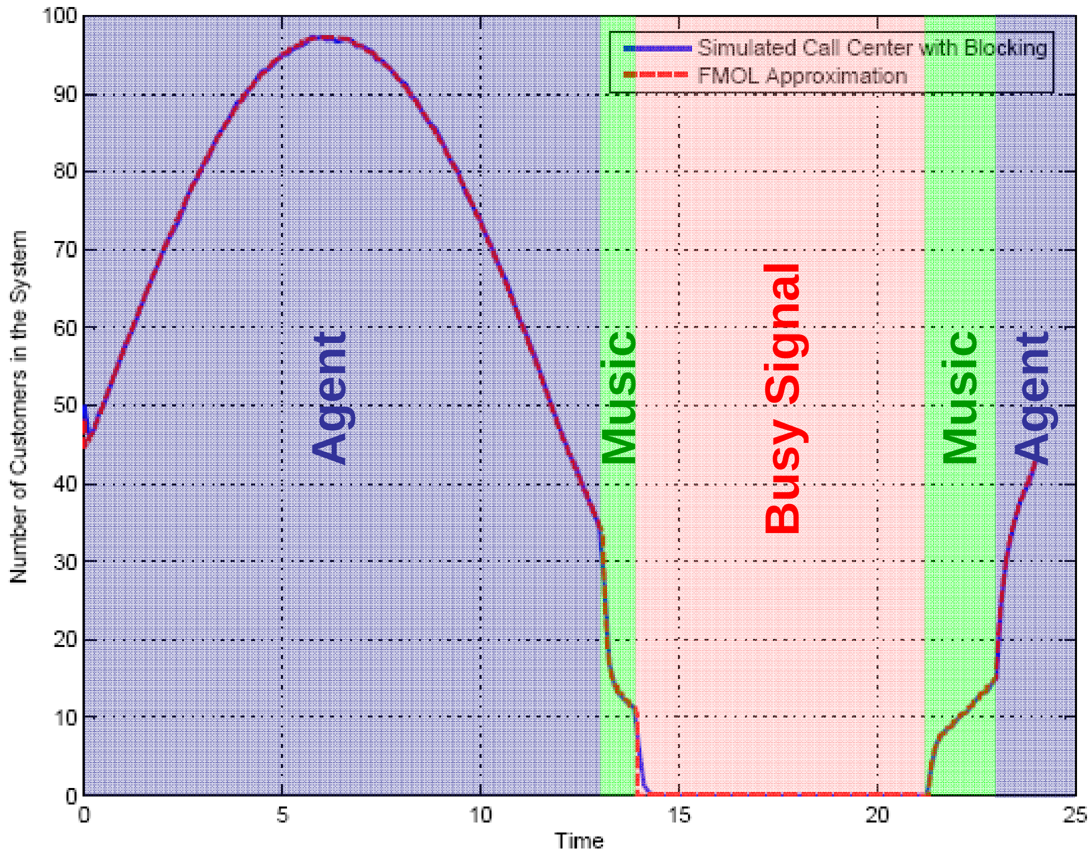
$$r = 1$$

$$\sigma = 0.10, \tau = 0.12 \quad \Leftrightarrow \quad \varepsilon_a = 0.10, \varepsilon_b = 0.05,$$

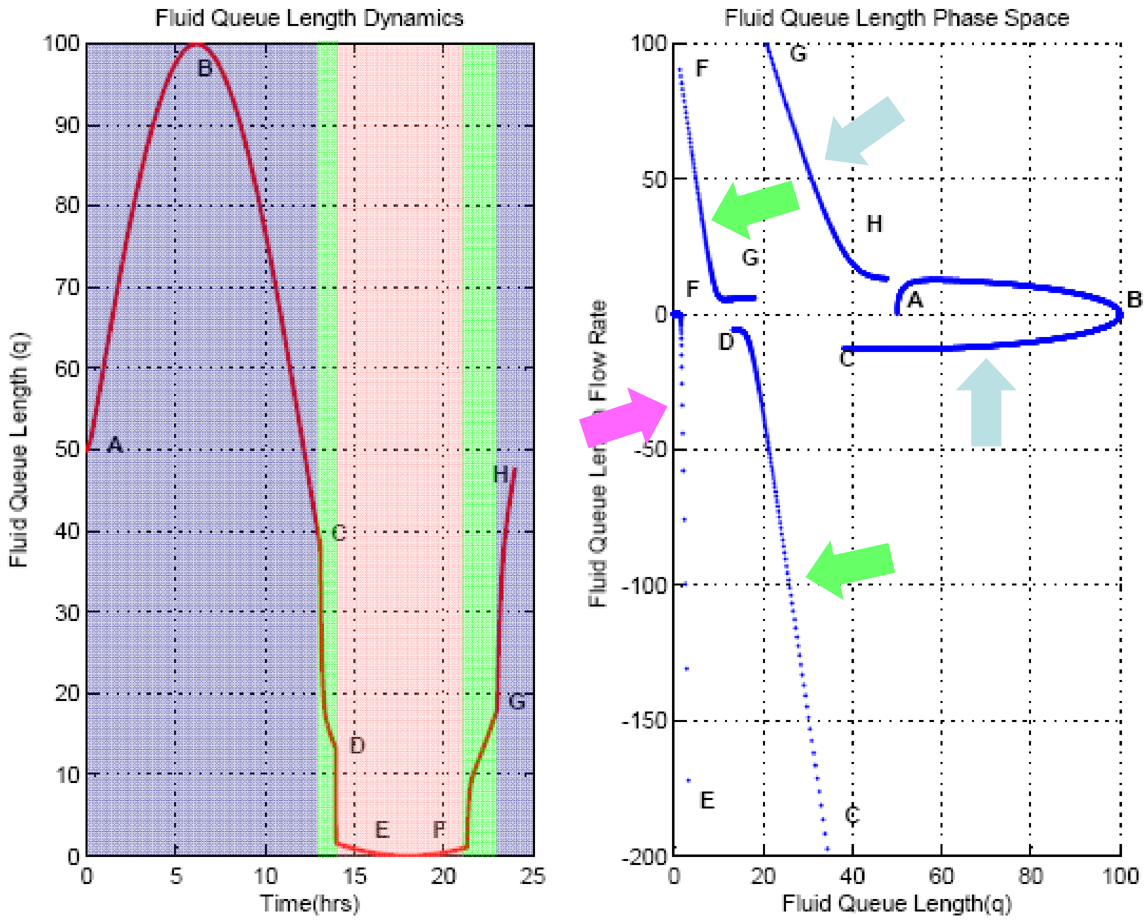
$$c(x) = 300 \cdot \log\left(1 + x \cdot 0.02 \cdot (e - 1)\right)$$

$$d(x) = \log\left(1 + x\right)$$

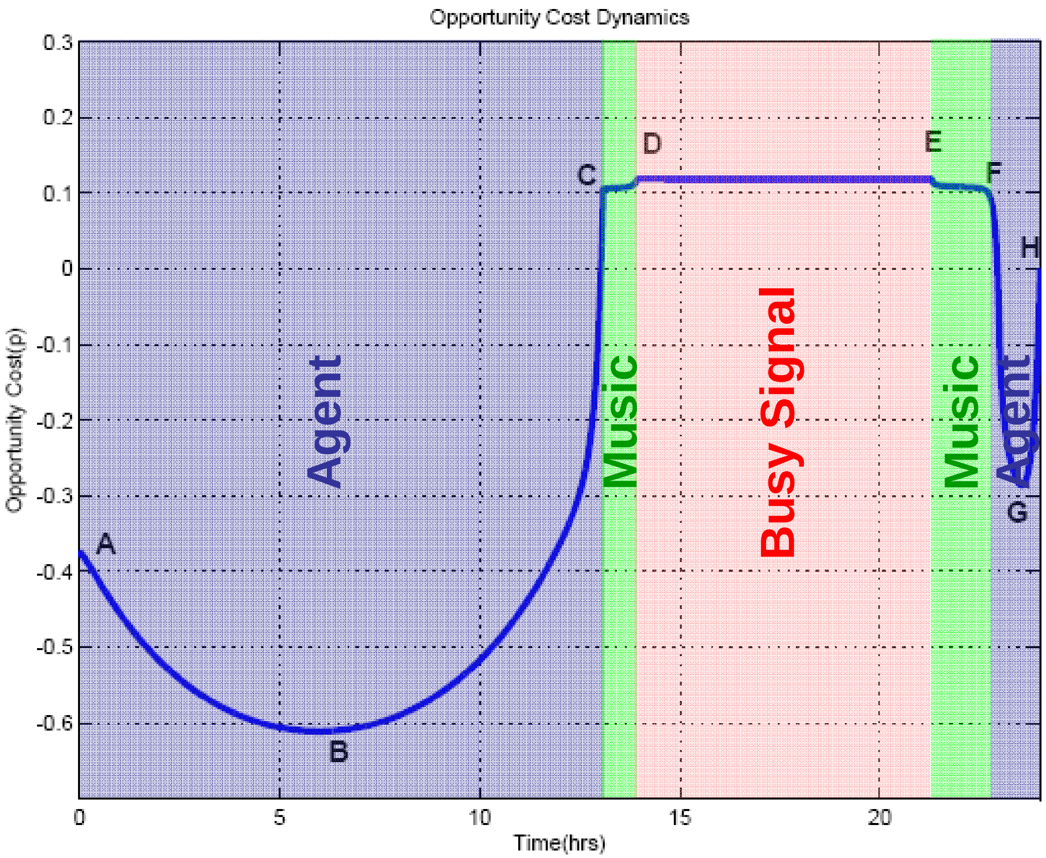
Comparison of Simulated Queue to FMOL Approximation



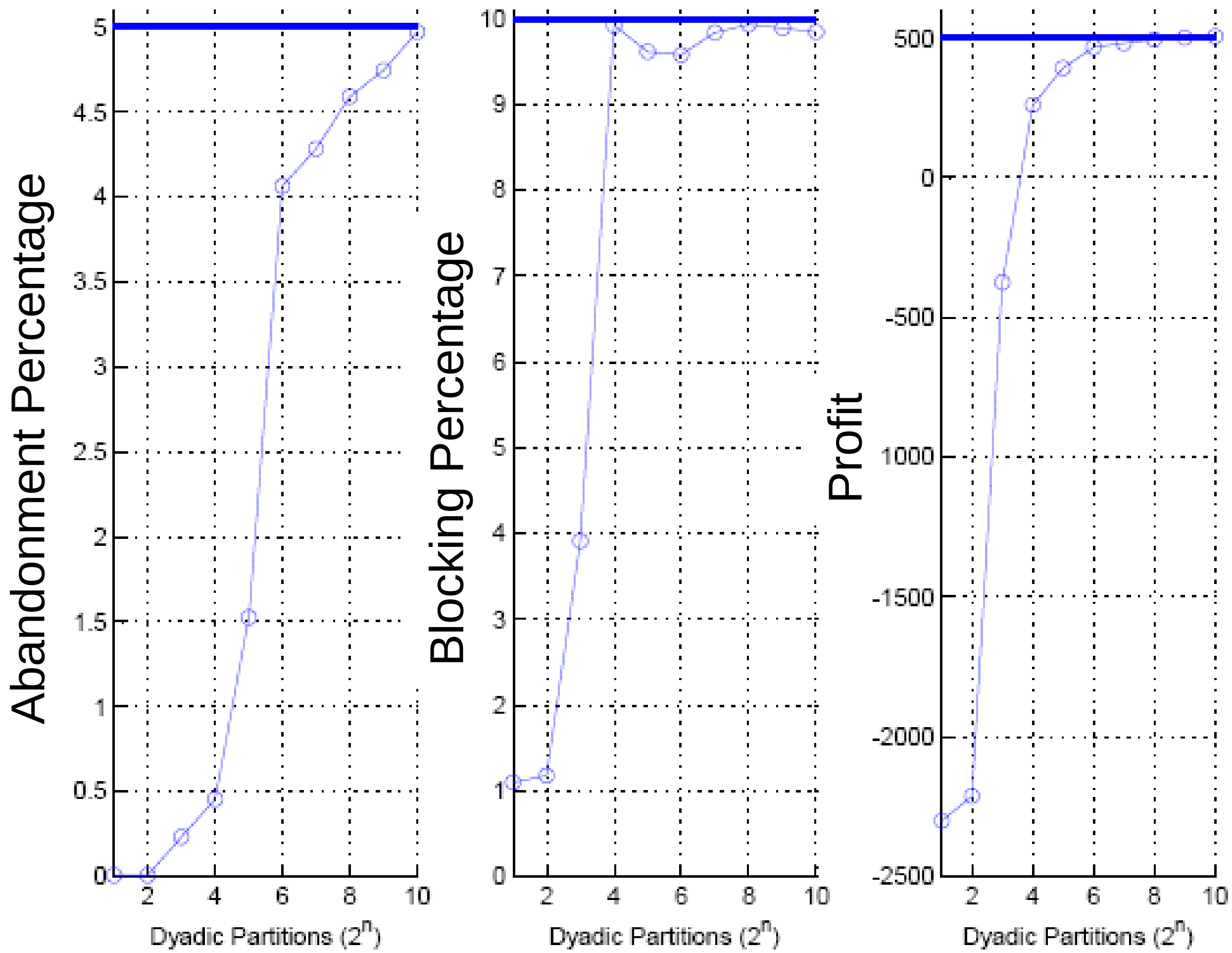
Time and Phase Plots of the Fluid Queue Length (*q*)



Plotting the Fluid Opportunity Cost (*p*)



Profit from min K - max L dyadic FMOL partitions improves monotonically.



Position	$q(t)$	Total Number of Customers
Velocity	$\dot{q}(t)$	Customer Flow Rate
Lagrangian	$\mathcal{L}(q,\dot{q},t)$	Value Rate
Generalized Momentum	$p(t)$	Opportunity Cost per Customer
Hamiltonian	$\mathcal{H}(p,q,t)$	Opportunity Cost Rate
Action	$\mathcal{V}(q,t)$	Bellman Value Function