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Assumptions*

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Extracting Probabilistic Information from the Prices of Interest Rate Options: Tests of Distributional Assumptions

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Abstract

It has been observed that return distributions in general and interest rates in particular exhibit skewness and kurtosis that cannot be explained by the lognormal distribution commonly used as an assumption in many option pricing models.

We have replaced the lognormal assumption in the Black (1976) model with the g-and-h distribution and derived a simple, closed-form option pricing formula under the no-arbitrage framework for pricing European options. We measured its performance using interest rate cap data and compared it with the option prices based on the Lognormal, Burr-3, Weibull, and GB2 distributions.

We observed that the g-and-h distribution exhibited a high degree of accuracy in pricing options and was found to be much better than these other distributions in extracting probabilistic information from the option market.

Key Words: g-and-h, GB2, Burr (type-3), Lognormal, Weibull, Interest Rate Cap, and Option Prices.

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Introduction

The options markets provide information on market expectations concerning the probability distribution of the underlying asset or instrument. Option pricing models can be used to infer these market expectations regarding the end-of-period probability distribution of the underlying asset or instrument and to estimate its parameter values based on observed prices, because option prices themselves are linked to pricing models featuring assumed probability distributions. We will call this an option-implied distribution. Black (1976) is a widely used model for the options we will price here. Therefore, in our application the benchmark model is the Black model. The distribution assumed in the Black model is the lognormal distribution. For the instruments we will be using,¹ prices are quoted in terms of the *ex ante* standard deviation (known as the implied volatility) of the lognormal distribution. Under the Black model one obtains different implied volatilities for different strike prices even when the option expiry date is the same, giving rise to the so-called “volatility smile.”

Dutta and Babbel (2002) observed that 1-month and 3-month-LIBOR do not conform to a lognormal distribution and that the skewness and kurtosis in the rates can be modeled effectively by some “flexible” leptokurtic distribution. Hawkin, Rubinstein and Daniell (1996), Sherrick et al (1996), Jondeau and Rockinger (2000), Navatte and Villa (2000), and Bahra (2001) are examples of some of the studies where the implied lognormal distributional assumption of the underlying asset or instrument has been empirically rejected. After Rubinstein (1994) had made a powerful argument for using general distributions in pricing the options, there have been some efforts to recover the probabilistic information implied by option prices using general distributions. In this effort, GB2 (McDonald (1991)), Burr type 3 (hereafter Burr-3, see Sherrick et al (1996)), and Weibull (Savickas (2001)) are some of the distributions that have been used to price options on various assets. In these and other studies, GB2 (McDonald (1991)) was the most general distribution and the very first attempt to price an option using a general distribution. However, only one of these attempts has been in the area of interest rate options. The closest was Rebonato (1999) who used GB2 to price a DEM cap. However, Rebonato offered no comparison of pricing accuracy with other distributional assumptions.

Guided by the evidence of the distributional properties of the 1-month and 3-month LIBOR as noted in Dutta and Babbel (2002), we will price interest rate options on this instrument with a g-and-h distribution and compare it with other (lognormal, Weibull, Burr-3, and GB2) distributional assumptions in extracting the probability distribution from the option market. We will consider here the European interest rate options. Our choice is influenced by the size and the nature of the products we will use. Because the US dollar interest rate cap is one of the most liquid interest rate options available on LIBOR, we will use it for the purpose of evaluating the performance of our distributional assumptions.

First, we will develop the necessary framework for option pricing. Second, we will price interest rate caplets under the assumption of g-and-h and other distributions. Finally, we will compare the performance of several models in extracting the implied distribution.

¹ In the foreign exchange and interest rate markets an implied volatility is quoted, whereas in the equity market actual price is quoted. The model used in both cases is the Black-Scholes model, which is slightly different from the Black (1976) model.

Framework for option prices

In order to price interest rate options based on different distributional assumptions, we first develop the necessary framework for option pricing. Babbel and Merrill (1996), Duffie (1996), Hull (2000), and Brigo and Mercurio (2001) are among the sources that provide comprehensive coverage on this subject.

Under general asset pricing theory with the no-arbitrage condition, the current price of an asset is equal to the present value of its expected payoffs discounted at an appropriate rate. The expectation can be made under any distributional assumption. Other than that the distribution needs to assume positive values,² there is no known economic theory that can be used to justify any particular assumption. We start with Black's model and show why it is theoretically consistent with the general asset pricing theory in pricing a wide variety of European interest rate options.

Black's model calculates the expected payoff from the option assuming

- (i) The underlying variable X is lognormally distributed at the expiry of the option with standard deviation of $\sigma\sqrt{T}$, where T is the time to maturity.
- (ii) The expected value of X at the maturity of the option is the forward value (F_0) of X at time 0, the valuation date.³

This expected payoff is then discounted by the T -duration risk-free rate at time zero. If we are pricing the call option then the payoff from the option is $\max(X_T - c, 0)$ at time T , where X_T is the value of X at time T and c is the strike rate of the option. The price of the call option is:

$$e^{-rT} \int_0^{\infty} \max(X_T - c, 0) f(X_T) d(X_T), \quad (1)$$

where $f(\cdot)$ is the density function of the lognormal distribution in Black's model. By evaluating the integral in (1) we obtain⁴

$$e^{-rT} (F_0 N(d_1) - c N(d_2)), \quad (2)$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal, $E(X_T) = F_0$, $d_1 = \frac{\ln(F_0/c) + \sigma^2 T/2}{\sigma\sqrt{T}}$, and $d_2 = d_1 - \sigma\sqrt{T}$

In the above derivation we have assumed that the interest rate is either constant or deterministic. Therefore, when the interest rate is stochastic, Black's model may appear to have made approximations in terms of 1) the behavior of the interest rate and 2) in assuming that $E(X_T) = F_0$. In a risk-neutral world where the interest rate is stochastic, $E(X_T) = \tilde{F}_0 \neq F_0$, where \tilde{F}_0 is the future price of X at time 0. However, as shown in Chapter 19 and 20 of Hull (2000), using the equivalent martingale measure in the world which is forward risk-neutral with respect to a zero-coupon bond maturing at time T , the two approximations have precisely offsetting effects when Black's model is applied to value bond options, interest rate caps/floors, and swap-

² If the underlying instrument is return on an asset then the distribution can assume negative values. There are some exotic options written on asset returns.

³ The valuation date and time zero will be used interchangeably.

⁴ Chapter 20 in Hull (2000) gives the computation for the integral.

tions. Therefore, when valuing these instruments, Black's model indeed has a strong theoretical basis and ensures arbitrage-free pricing.

The lognormal density assumption in Black's model is an arbitrary assumption and is not required to price an option in the risk-neutral world. On the contrary, there is strong evidence that the underlying asset or instrument is often not lognormally distributed. Therefore, we will replace the lognormal assumption in Black's model with various distributions and test our assumptions. We will use interest rate caplets to estimate and evaluate the parameters.

The interest rate cap is one of the most liquid interest rate options traded in the market. It is comprised of a portfolio of caplets. A caplet is an interest rate call option on short-term interest rates whose strike rate is the cap rate. A caplet's payoff is based on the level of the reference interest rate on the date of the caplet's maturity but the payment is generally made in arrears.⁵ The price of the cap is equal to sum of the prices of the caplets.

Suppose a cap is written on the principal amount P (known also as "notional"), with strike rate c , and for a total duration of time T (known as tenor). Let the entire time period T be partitioned into $t_0, t_1, t_2, \dots, t_n, t_{n+1} = T$. The intermediate points in the partition are known as the reset dates. Suppose the interest rates on $t_0, t_1, t_2, \dots, t_n$ are r_1, r_2, \dots, r_n . $r_i (1 \leq i \leq n)$ are the interest rates for the periods between t_i and $t_{i+1} (1 \leq i \leq n-1)$ observed at time t_i . The caps are priced in such a way that there is no loss or gain at time t_0 . For each period t_i to t_{i+1} there is a caplet that matures at time t_i and settles (transactions made) at time t_{i+1} . The amount transacted at t_{i+1} is $P\delta_{t_i} \max(r_i - c, 0)$, where δ_{t_i} is the compounding factor⁶ for the period from t_i to t_{i+1} . Since the transaction is not made until the next period, in pricing a caplet the discounting factor in equation (1) should be adjusted accordingly. The market price of a cap (and hence of a caplet) is quoted on a notional of one currency unit. Therefore, the price of a caplet valued at time t_0 , maturing at time t_i , and settling at time t_{i+1} , on a notional value of one currency unit is:

$$e^{-rt_{i+1}} \delta_{t_i} \int_0^{\infty} \max(X_{t_i} - c, 0) f(X_{t_i}) d(X_{t_i}), \quad (3)$$

where X_{t_i} is the interest rate at t_i , $f(\cdot)$ is the density function (not necessarily lognormal) of X_{t_i} , and r is the risk-free rate for the period between t_0 and t_{i+1} . Also as before, we are assuming our economy is forward risk-neutral. Therefore

$$E(X_{t_i}) = F_{t_0, t_i}, \quad (4)$$

where F_{t_0, t_i} is the forward interest rate for the period from t_0 to t_i at time t_0 .

As we have seen before, the interest rate caplets are essentially an interest rate option for the period from t_0 to t_i and one of the most liquid options on interest rates. Therefore either to extract the probabilistic information or to test the distributional assumptions of the short-term interest rate, the caplets will be the best instrument. In particular, we will be using the US dollar caplets for our analysis here. With this necessary framework developed in this section, we can now price the option under the different distributional assumptions.

⁵ Not all caps pay in arrears. See in Merrill and Babbel (1996).

⁶ Interest rates are quoted on an annualized basis.

Option Pricing under Different Distributional Assumptions

Since the main objective of this work is to recover the probabilistic information of the short-rate from the interest rate option market, we first price the options under different distributional assumptions. We will provide in detail the option pricing formula using the g-and-h distribution since we know of no published work where this has been done. For other distributions we will refer to and adopt the pricing formulae given in published works. A more generalized treatment of valuation and risk management techniques across many markets and instruments is provided in Dutta (2002).

Option pricing with g-and-h distribution

The g-and-h distribution was introduced by Tukey (1977). Martinez and Iglewicz (1984), Hoaglin (1985), Badrinath and Chatterjee (1988 and 1991), Mills (1995), and Dutta and Babbal (2002) also studied the properties of this distribution. Badrinath and Chatterjee, and Mills used the g-and-h distribution to model the return on equity indices in various markets, whereas Dutta and Babbal used it to model LIBOR rates. Tukey introduced a family of distributions by transforming the standard normal variable Z to

$$Y_{g,h}(Z) = (e^{gZ} - 1) \frac{\exp(hZ^2/2)}{g},$$

where g and h are any real numbers. By introducing location (A) and scale (B) parameters, the g-and-h distribution has four parameters in the following form:

$$X_{g,h}(Z) = A + B(e^{gZ} - 1) \frac{\exp(hZ^2/2)}{g} = A + BY_{g,h} \quad (5)$$

When $h=0$, the g-and-h distribution reduces to $X_{g,0}(Z) = A + B \frac{(e^{gZ} - 1)}{g}$, which is also known as the g-distribution. The g parameter is responsible for the skewness of the g-and-h distribution. The g-distribution exhibits skewness but no kurtosis.

Similarly when $g=0$, the g-and-h distribution reduces to

$$X_{0,h}(Z) = A + BZ \exp(hZ^2/2) = A + BY_{0,h}, \quad (6)$$

which is also known as the h-distribution. The h parameter in g-and-h distribution is responsible for its kurtosis. The h-distribution has fat tails (kurtosis) but no skewness. As noted in Martinez and Iglewicz (1984), many commonly used distributions can be derived as a special case of the g-and-h distribution.

To price the call option using a g-and-h distribution we need to evaluate the integral in step (1) with the g-and-h density in (5). The integral in step (1) is equivalent to $E[\max((X_T - c), 0)]$. If X_T follows a g-and-h distribution then $X_T = a + \frac{b(e^{gZ} - 1)e^{hZ^2/2}}{g}$, where Z is a standard normal distribution. Therefore,

$$E[\max((X_T - c), 0)] = \frac{1}{\sqrt{2\pi}} \int_c^\infty \left(a + \frac{b(e^{gZ} - 1)e^{hZ^2/2}}{g} - c \right) e^{-Z^2/2} dZ \quad (7)$$

Equation (7) can be split into three parts as follows.

$$1) \quad \frac{1}{\sqrt{2\pi}} \int_c^\infty (a - c) e^{-Z^2/2} dZ = (a - c)(1 - N(c)) \quad (8)$$

$$2) \quad \frac{-1}{\sqrt{2\pi}} \int_c^\infty \left(b \frac{e^{-(1-h)Z^2/2}}{g} \right) dZ \quad (9)$$

Transforming $Z \rightarrow y/\sqrt{1-h}$ the integral in (9) becomes

$$\frac{1}{\sqrt{2\pi}} \int_{c\sqrt{1-h}}^\infty b \frac{e^{-y^2/2}}{g(\sqrt{1-h})} dy = \frac{-b}{g(\sqrt{1-h})} (1 - N(c\sqrt{1-h})) \quad (10)$$

$$3) \quad \frac{b}{g\sqrt{2\pi}} \int_c^\infty e^{-\left(\frac{(1-h)Z^2}{2} - gZ\right)} dZ \quad (11)$$

Completing the square in the exponent of e in (11) we have

$$\frac{b}{g\sqrt{2\pi}} e^{\frac{g^2}{2(1-h)}} \int_c^\infty e^{-\frac{1}{2}\left(\sqrt{1-h}Z - \frac{g}{\sqrt{1-h}}\right)^2} dZ \quad (12)$$

By making the transformation $y \rightarrow \sqrt{1-h}Z - \frac{g}{\sqrt{1-h}}$ to the integral in (12) we have

$$\frac{b}{g\sqrt{1-h}} e^{\frac{g^2}{2(1-h)}} \frac{1}{\sqrt{2\pi}} \int_{c\sqrt{1-h} - \frac{g}{\sqrt{1-h}}}^\infty e^{-\frac{1}{2}y^2} dZ = \frac{b}{g\sqrt{1-h}} e^{\frac{g^2}{2(1-h)}} (1 - N(c\sqrt{1-h} - \frac{g}{\sqrt{1-h}})) \quad (13)$$

Combining (8), (10), and (13) we get the call price using the g-and-h distribution as

$$e^{-rT} E[\max(X_T - c, 0)] = e^{-rT} \left\{ (a - c)(1 - N(c)) - \frac{b}{g\sqrt{1-h}} [1 - N(c\sqrt{1-h})] + \frac{b}{g\sqrt{1-h}} e^{\frac{g^2}{2(1-h)}} [1 - N(c\sqrt{1-h} - \frac{g}{\sqrt{1-h}})] \right\} \quad (14)$$

where

$$E(X_T) = a + \frac{b(e^{\frac{g^2}{2(1-h)}} - 1)}{g\sqrt{1-h}} = F_0. \quad (15)$$

We can eliminate a parameter between equations (14) and (15) and express equation (14) in terms of F_0 the forward price of X_T on the valuation date.

Similarly we price the put option as follows.

$$E[\max(p - X_T, 0)] = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^p \left(p - a + \frac{b(e^{gZ} - 1)e^{hZ^2/2}}{g} \right) e^{-Z^2/2} dZ \text{ where } p \text{ is the strike price for a put.}$$

Following the steps as before we get the price of the put as:

$$e^{-rT} \left[(p-a)N(p) + \frac{b}{g\sqrt{1-h}} N(p\sqrt{1-h}) - \frac{b}{g\sqrt{1-h}} e^{\frac{g^2}{2(1-h)}} N(p\sqrt{1-h} - g/\sqrt{1-h}) \right] \quad (16)$$

Putting $p = c = X$ and subtracting (16) from (14) we get:

$$e^{-rT} (F_0 - X) \quad (17)$$

From step (17) we conclude that our option prices using the g-and-h distribution have preserved put-call parity, a necessary relationship to validate any option pricing formula. To price a caplet we multiply the call price in (14) with δ_{t_0} , the compounding factor.

Option Pricing with Generalized Beta Distribution of Second Kind (GB2)

The Generalized Beta Distribution of the Second Kind (GB2), like the g-and-h distribution, can accommodate a wide variety of tail-thickness and permits skewness as well. Bookstaber and McDonald (1987), McDonald (1991 and 1996), and McDonald and Xu (1995) have analyzed the properties and applications of the GB2 distribution in detail. Bookstaber and McDonald (1987), and McDonald (1996) have explored the possibility of modeling asset returns using GB2. GB2 distribution is defined as:

$$GB2(y; a, b, p, q) = \frac{|a| y^{ap-1}}{b^{ap} B(p, q) [1 + (y/b)^a]^{p+q}} \text{ when } y > 0, \quad (18)$$

$$= 0 \text{ otherwise}$$

Here, $B(p, q)$ is a Beta function. Like the g-and-h, GB2 is a four-parameter distribution. Some of the useful properties of GB2 are summarized below.

The cumulative distribution function of GB2 is given by⁷

$$X(y; a, b, p, q) = z^p {}_1F_2[p, 1-q, 1+p; z] / pB(p, q) \quad (18a)$$

where $z = (y/b)^a / (1 + (y/b)^a)$ and ${}_1F_2[a, b, c, d]$ is a hypergeometric function.⁸ Bookstaber and McDonald (1987) noted that many commonly used distributions can also be derived as a special case of GB2.

McDonald (1991) developed a method to price options using the GB2 distribution. The method adopted by McDonald is based on normalized incomplete moments. The h th normalized incomplete moment of a distribution is defined as $\varphi(y; h) = \frac{\int_0^y s^h f(s) ds}{E(Y^h)}$. The complement of the

normalized incomplete moment is $\bar{\varphi} = 1 - \varphi$. From the density function of GB2 we can infer that computing the option prices using the expression in (1) will be quite cumbersome.

Rebonato (1999) used the method developed by McDonald and computed the call option price⁹ based on GB2, which is equal to:

⁷ For derivation see McDonald and Xu (1995) and McDonald (1996).

⁸ Hypergeometric is a special function. Reference for such functions will be Abramowitz and Stegun (1972).

⁹ For the price of a put see Rebonato (1999).

$$e^{-rT} \left(\frac{X \left(\frac{b}{X}\right)^{aq} {}_1F_2 \left[q - \frac{1}{a}, p + q, 1 + q - \frac{1}{a}, -\left(\frac{b}{X}\right)^a \right]}{\left(q - \frac{1}{a}\right) B(p, q)} - \frac{X \left(\frac{b}{X}\right)^{aq} {}_1F_2 \left[q, p + q, 1 + q, -\left(\frac{b}{X}\right)^a \right]}{q B(p, q)} \right) \quad (19)$$

where X is the strike rate and ${}_1F_2[j, k, l, m]$ is the hypergeometric function. Multiplying the compounding factor as before, we get the price of the caplet. The call price given in (19) is slightly different from the one given in Rebonato (1999). Rebonato assumed a zero interest rate economy for discounting purposes, which is not a realistic assumption. As before, the following condition should also hold to satisfy the risk neutrality of the forward prices:

$$E(X_T) = \frac{b B\left(p + \frac{1}{a}, q - \frac{1}{a}\right)}{B(p, q)} = F_0 \quad (20)$$

Using equations (19) and (20) we can eliminate one parameter out of the four parameters of the GB2 distribution and express the call price in (19) in terms of F_0 . Rebonato has equated the first and second moments of the GB2 distribution with those of the lognormal and called it an equivalent volatility. By doing so one can eliminate one more parameter of GB2 in the option price given in (19). There is no known empirical or economic justification for this approach. In that assumption, one would give up the flexibility of the distribution.

Option Pricing with Other Distributions

We also used Burr-3 and the Weibull distributions to compare the option prices based on g-and-h and GB2 distributions.

Burr-3 Distribution

The Burr-3 distribution is a special case of the GB2 distribution (Bookstaber and McDonald, 1987). When the parameter $q = 1$ in (18) we have the Burr-3 distribution:

$$\begin{aligned} \text{Burr3}(y; a, b, p) &= \frac{|a| y^{ap-1}}{b^{ap} B(p, 1) [1 + (y/b)^a]^{p+1}} \text{ when } y > 0, \\ &= 0 \text{ otherwise} \end{aligned}$$

Therefore, Burr-3 is a distribution with three parameters. Sherrick *et al* (1996) used it to price options on soybean futures. Even though Burr-3 has received limited attention in modeling asset returns, it has been found to be useful in describing the loss distribution in the insurance industry.

Since Burr-3 is a special case of GB2 ($q = 1$), therefore substituting for the value of q in (19) and (20) we get the price of a call option (and caplet) using the Burr-3 distribution. The option price associated with the Burr-3 distribution has only two free parameters.

Weibull Distribution

The Weibull distribution can also be derived as a limiting case from g-and-h as well as from GB2. The Weibull distribution belongs to the family of distributions known as Extreme-Value distributions. The density function of the Weibull distribution is given by

$$f(x) = abx^{b-1} e^{-ax^b}, \quad a > 0, \quad b > 0, \quad \text{and } x > 0$$

Weibull is therefore a distribution with two parameters. Savickas (2001) describes the properties of the Weibull distribution and compares it with the lognormal distribution.

Savickas has developed the option price under the Weibull distribution using (1). Substituting the Weibull density in (1) we obtain the price of a call option:

$$e^{-rT} \int_c^{\infty} (X_T - c) \alpha \beta X_T^{\beta-1} e^{-\alpha X_T^\beta} dX_T \quad (21)$$

After simplification (21) reduces to

$$e^{-rT} \left[\frac{\Gamma(1+1/\beta)}{\alpha^{1/\beta}} \left(1 - \frac{\Gamma_\omega(1+1/\beta)}{\Gamma(1+1/\beta)} \right) - c e^{-\omega} \right] \quad (22)$$

where $\omega = \alpha c^\beta$ and $\Gamma_\omega(1+1/\beta) = \int_0^\omega x^{1/\beta} e^{-x} dx$ is the incomplete gamma function (see Abramowitz and Stegun (1964)). Substituting $E(X) = \frac{\Gamma(1+1/\beta)}{\alpha^{1/\beta}} = F_0$ in (22), we have

$$e^{-rT} \left[F_0 \left(1 - \frac{\Gamma_\omega(1+1/\beta)}{\Gamma(1+1/\beta)} \right) - c e^{-\omega} \right] \quad (23)$$

As before, multiplying (23) by the compounding factor, we get the price of the caplet using the Weibull distribution.

We now have all the caplet prices we need to test the distributional assumptions implied by option prices. In the next section we will first estimate the parameter and then test the assumptions with the caplet data.

Tests of the Distributional Assumptions

In order to test the distributional assumption, we first need to estimate the parameters of the distribution. The caplet prices calculated in earlier section will be used to estimate the parameters. Let the model price of the caplet be E_Δ^i where Δ is the set of parameters to be estimated. Let O_i be the market prices of the caplet. Let n be the total number of caplets used to estimate the parameters. Then the best estimate of the Δ is Δ_{\min} , the parameter set that minimizes

$$\sum_{i=1}^n (O_i - E_\Delta^i)^2 \quad (24)$$

subject to the $E(X_T) = F_0$ and other parameter constraints specific to a particular distribution.

The optimization problem described in (24) is a nonlinear optimization problem, which adds several complexities in estimating the parameters. We have encountered situations where the solution did not exist. We used the optimizer (solver)¹⁰ in Microsoft Excel to solve the problems.

¹⁰ The solver was enhanced by the Premium Solver Platform, software obtained from the Frontline Systems. We tried to solve the optimization problems by the MATLAB optimizer as well. We found that the Premium Solver Platform performed better than the MATLAB for our application.

Data Description

According to the International Swaps and Derivatives Association, the total notional principal amount of over-the-counter US dollar interest rate options such as caps/floors and swaptions exceeded \$6 trillion at the end of 2000. This amount was more than 50 times the \$120 billion in combined notional principal of all the options on Treasury notes and bond futures traded at the Chicago Board of Trade. Therefore, caps/floors are one of the most liquid interest rate options that can be used to infer an implied probability distribution. As we explained earlier, the liquidity of the option is important for it to be used in recovering the probability distribution.

The US dollar caps are quoted in basis points.¹¹ The price of the contract is multiplied by the notional principal amount to give the dollar value of the contract. Since the caps are over-the-counter traded contracts, the data relating to caps are available only from the broker/dealer or market maker. Most of the available data are quoted on an at-the-money forward basis, which means that on any trading day and for one specific tenor the quote of only one strike is available. We noted earlier that as we move away (in either direction) from an at-the-money forward, the option starts exhibiting the volatility smiles. The trading strategy of caps as well as of many other interest rate instruments is based on this volatility smile.

We obtained end of the day closing prices for US dollar caps of different strikes and tenors from Gerban Intercapital, a major broker/dealer and market maker in interest rate caps/floors and swaptions. The tenors of the cap were of six different maturities (2, 3, 4, 5, 6, and 7-years) and of eight different strikes (5%, 5.5%, 6%, 6.5%, 7%, 7.5%, and 8%). However, caps at all these strikes were not always quoted for each of the maturities. The sample period consisted of 141 trading days¹² of daily data from October 23, 2000 to September 19, 2001. In total, 3,769 contracts were used for the estimation of the parameters. The liquidity of the contracts on a given day varied according to the strikes, maturities, and the 3-month-LIBOR rate on that day. Deep in-the-money as well as deep out-of-the-money caps exhibited less liquidity. We have disregarded any quote with an open interest less than 10. Also, options with short maturities (less than 3 years) exhibited liquidity only for at-the-money-forward strikes. Therefore, even though we could obtain the data for a 1- year cap we chose not to use it due to the lack of its liquidity. Table 1 gives the basic statistics of the cap data we used. From the cap prices we obtained the caplet prices.¹³ For the purpose of computing these caplet prices we needed the forward (and discount) curve(s). We used 1, 3, 6, and 12-month LIBOR, 2, 3, 5, 7, and 10-year US dollar interest rate swap data to construct the forward curve (and discount curve) for each trading day. This discount curve was used in our option valuation models as well. Using these caplet prices we estimated the parameters of the implied distributions for each of the option models discussed earlier. We used the caplets that matured at the end of the ninth month and settled at the end of twelfth month from the beginning of the cap.

¹¹ The prices are normally quoted in terms of implied volatilities. However, we obtained them in basis points.

¹² There were approximately 252 trading days. However, on approximately 100 trading days there was no noticeable price movement from the previous day. In our analysis we used distinct prices to estimate the parameters.

¹³ We used FINCAD tools to compute the prices.

The Testing Methodology and Model Evaluation

The testing methodology we adopted here is similar to the ones in Jackwerth and Rubinstein (1995), Martinez (1998), Buhler *et al* (1999), Bali (2000), Driessen *et al* (2000), Gupta and Subrahmanyam (2001), and Savickas (2001). Caplets were classified based on tenors. For each tenor the parameters of the distributions were estimated using the methodology outlined in (24). The significance of this classification lies in testing the stability of the parameters and their out of sample performance.

Using the estimated parameters, the caplet prices were computed under each of the different distributional assumptions. The (percentage) errors between the market and the model prices (both relative and absolute) were computed. For each tenor the average was computed by taking all caplets across different strikes and for all the trading days in our sample. Table 2 shows the average (percentage) errors (both absolute and relative) across different maturities for the option prices under various distributional assumptions.

In addition to estimating the errors, the model performance was also evaluated by the following statistical tests.

- (i) The model price and market price are regressed by the following regression equation: $market\ price_i = \beta_{i0} + \beta_{i1} + model\ price_i + \varepsilon_i$, and the values of the coefficients, standard errors, and the R^2 s are noted.
- (ii) The correlation coefficients of the errors and percentage errors (both relative and absolute) were computed among the different models and for all maturities.
- (iii) The basic statistics (mean, standard deviation, median, percentile etc) for the parameter estimates were obtained across all maturities.

From Table 2 we can see that on the basis of both relative and absolute errors, the g-and-h distribution exhibited the highest accuracy in extracting the implied distribution. The minimum and maximum absolute average errors obtained were 4.62% and 9.88%, respectively. The average percentage relative and absolute errors from using the g-and-h distribution are significantly smaller than the errors from other distributions. Although insignificant, the negative value of the average relative error (consistently obtained across all maturities) indicates that g-and-h, on average, over priced the options. The GB2 distribution showed the next best accuracy. However on a percentage basis, GB2 exhibited much higher inaccuracies than g-and-h. In certain instances, the solution for (24) did not exist for the GB2 distribution. Therefore, in those instances GB2 violated arbitrage-free pricing. Like g-and-h, GB2 also over priced the options consistently but at a much higher differences than g-and-h. The minimum and maximum absolute average errors were 16.9% and 51.75%, respectively. Figure 1 shows graphically the relative errors for different maturities under different distributions. For other distributions (Burr-3, lognormal, and Weibull) the errors were extremely high indicating that the implied distribution is different from Burr-3, lognormal, and Weibull. Also, we observed that the errors decreased from shorter maturities to longer maturities. One possible reason for this is that liquidity increased from shorter maturities to longer maturities. We had more contracts to solve (24) for the longer maturities than for the shorter maturities.

Tables 3 and 4 show the results of the regression statistic for g-and-h and GB2 distributions, respectively. From the *Rsquare* column in the tables we conclude that there is a high de-

gree of correlation between market and model prices under the g-and-h distribution and in many instances under the GB2 distribution as well. While we observed no value of R^2 less than 82% for the g-and-h distribution, we observed several R^2 values for GB2 less than 60%. For the Burr-3, lognormal and Weibull distributions we often observed very low correlations between market and model prices (Tables 5, 6 and 7).

Figure 2 shows the average values of the implied parameter estimates of g-and-h for all trading days in our sample. We observed high volatilities in the parameter estimates for g-and-h. This is consistent with the g-and-h parameter estimates that Dutta and Babbel (2002) obtained using the historical 1-month and 3-month-LIBOR data. Therefore, we observe that both historical as well as implied estimates of the parameters of g-and-h show high volatility. Similar observations were obtained for the GB2 distribution (Figure 3).

We observed a very high degree of inaccuracy between the model and the market prices under the assumptions of Burr-3, Lognormal, and Weibull distributions. Table 8 shows the correlation coefficients of errors between different distributional assumptions. From Table 8, we can see that there is positive correlation between the g-and-h and GB2 distributions. The highest and lowest correlation coefficients between these two distributions were 0.69 and 0.45, respectively. With respect to the g-and-h distribution, we observed virtually no correlations with the Burr-3 and Weibull distributions. For shorter maturities the g-and-h distribution showed negative correlation with the lognormal distribution, but at longer maturities the correlation coefficient was positive and of approximately the same value as was with the GB2.

Conclusion

Dutta and Babbel (2002) observed that the skewed and leptokurtic behavior of LIBOR could be modeled effectively by the g-and-h distribution. The estimates they made can be viewed as backward-looking since it was based on what actually happened in the past. The market's expectation of the distributional properties of LIBOR can be extracted from option prices. Here we attempted to model the skewed and leptokurtic behavior of the 3-month LIBOR data as implied by its option prices. In that respect, the estimates made here could be thought of as forward-looking. We observed that the implied distribution of 3-month LIBOR could be modeled very accurately with the g-and-h distribution. Gupta and Subrahmanyam (2001) priced US dollar caps using many well-known term structure models and reported errors in many instances of a higher magnitude than what we obtained using the simple g-and-h distribution. In addition, the regression statistics along with the correlation of errors with other distributions signify an extremely good fit between the implied distribution of the 3-month LIBOR data and the g-and-h distribution. Therefore, we can conclude that the market expected 3-month LIBOR to be skewed and leptokurtic which can be modeled by the g-and-h distribution with a high degree of accuracy.

The GB2 distribution is also a general skewed and leptokurtic distribution and we have every reason to believe that we could have modeled the implied distribution with the GB2 just as accurately as with the g-and-h distribution. The inaccuracy we observed in GB2-based prices was probably due to the complexity involved in computing such prices as is evident from (19). Dutta and Babbel (2002) observed that the GB2 distribution is highly sensitive to its parameter values. Small changes in the parameter values may result in large differences in the option prices. The computational simplicity of the g-and-h distribution is definitely one of the reasons for the accuracy we observed in its prices over the GB2 distribution. Rebonato (1999) reported

very high degrees of accuracy in cap (caplet) prices of the DEM caplets using GB2. However, he reported the result for only one trading day.¹⁴

Even though some authors reported great success in modeling skewness and kurtosis by Burr-3 and Weibull, we did not observe a good fit for our application. These distributions with a restricted number of free parameters could not model the skewed and leptokurtic behavior of the 3-month LIBOR effectively. Based on the statistics observed, we conclude that the option implied distribution of the 3-month LIBOR is not lognormal either. The ample data on 3-month LIBOR options led us to focus on that tenor and instrument in our experiments. Neither in our development of the model nor in our testing did we assume any particular economic properties of 3-month-LIBOR. Therefore, we strongly believe that other short-rates can also be modeled effectively by the g-and-h distribution.

¹⁴ Rebonato (1999) claimed that they obtained similar results for many other trading days. It is not clear if the experiment was conducted for substantially longer periods as we did.

Table 1

The following table presents the descriptive statistics of the US Dollar interest rate caps used to estimate the parameters of the distributions. The prices of the contracts are expressed in basis points (1bp = 0.01%). The total number of contracts used for the estimation purpose was 3,769.

	5%						5.5%					
	2yr	3yr	4yr				2yr	3yr	4yr	5yr	6yr	7yr
Mean	54	139	235				36	110	201	291	381	478
Max	81	187	252				61	143	249	368	423	517
Min	33	120	228				20	79	157	241	339	433
Stdev	12	16	9				10	17	23	27	25	26
25% Percentile	44	126	230				29	96	182	272	360	458
75% Percentile	64	148	239				43	122	219	306	403	504
Median	54	136	231				36	109	197	286	379	482
Count	94	40	11				119	119	101	64	34	17
	6%						6.5%					
	2yr	3yr	4yr	5yr	6yr	7yr	2yr	3yr	4yr	5yr	6yr	7yr
Mean	22	76	148	232	320	410	27	67	109	167	232	308
Max	57	111	194	300	404	500	53	113	173	242	303	384
Min	12	53	113	179	252	332	9	32	73	125	182	248
Stdev	8	14	22	30	39	43	17	29	28	30	32	35
25% Percentile	16	64	127	206	288	370	13	45	87	151	217	288
75% Percentile	25	85	168	258	355	441	45	93	124	180	246	318
Median	20	73	143	229	320	407	20	52	101	157	223	306
Count	124	124	124	124	123	120	44	42	34	32	31	31
	7%						7.5%					
	2yr	3yr	4yr	5yr	6yr	7yr	2yr	3yr	4yr	5yr	6yr	7yr
Mean	9	36	78	132	191	253	7	25	56	99	146	202
Max	26	68	121	192	264	337	13	41	82	136	194	255
Min	2	15	44	86	131	182	2	8	27	58	94	138
Stdev	6	13	21	28	35	40	4	12	20	26	32	37
25% Percentile	5	27	61	111	165	229	3	14	38	76	121	175
75% Percentile	9	42	94	156	222	287	12	38	78	122	176	235
Median	6	32	75	131	189	250	7	28	62	98	146	194
Count	141	141	141	141	141	141	44	44	44	44	44	44
	8%						8.50%					
	2yr	3yr	4yr	5yr	6yr	7yr	2yr	3yr	4yr	5yr	6yr	7yr
Mean	3	15	38	70	108	150	2	10	26	52	83	120
Max	11	27	57	99	147	195	6	18	40	72	107	150
Min	1	4	16	39	66	100	1	2	10	27	47	73
Stdev	2	6	12	17	23	27	1	5	11	16	21	24
25% Percentile	1	10	26	56	91	132	1	5	16	38	68	102
75% Percentile	3	20	49	86	130	175	3	14	35	66	101	142
Median	2	15	39	69	108	150	2	12	31	60	89	123
Count	138	141	141	141	141	141	44	44	44	44	44	44

Table 2

The following table presents the summary statistics of the forecast errors (in basis points and percentages) for the **Lognormal**, **g-and-h**, **GB2**, **Burr-3** and **Weibull** distributions.

Model	Tenor	Average	Average	Average Abs	Average Abs
		Error(bp)	Error(%)	Error(bp)	Error(%)
lognormal	2Yr	3.459	-52.68%	5.667	107.32%
	3Yr	2.209	-67.46%	4.694	114.75%
	4Yr	-0.101	-443.35%	3.851	476.44%
	5Yr	-0.275	-563.64%	3.225	593.35%
	6Yr	-0.244	-532.79%	2.832	562.78%
	7Yr	-0.226	-533.82%	2.566	564.03%
	2Yr	-0.058	-12.16%	0.612	17.34%
	3Yr	-0.050	-12.20%	0.507	16.90%
GB2	4Yr	-0.138	-47.23%	0.379	50.64%
	5Yr	-0.099	-48.93%	0.258	51.75%
	6Yr	-0.072	-41.51%	0.185	44.14%
	7Yr	-0.056	-38.10%	0.136	40.54%
	2Yr	-0.038	-4.35%	0.189	6.07%
	3Yr	-0.034	-4.63%	0.159	6.02%
g-h	4Yr	-0.022	-8.57%	0.095	9.88%
	5Yr	-0.015	-6.49%	0.052	7.61%
	6Yr	-0.011	-4.91%	0.034	5.71%
	7Yr	-0.008	-3.70%	0.023	4.62%
	2Yr	4.013	-17.36%	5.047	74.09%
	3Yr	3.030	-25.13%	4.181	76.72%
Burr3	4Yr	5.175	99.32%	5.175	99.32%
	5Yr	0.404	-333.31%	2.360	359.85%
	6Yr	-0.018	-364.67%	2.092	387.03%
	7Yr	-0.336	-397.67%	1.903	416.91%
	2Yr	7.033	18.91%	11.478	69.52%
	3Yr	5.129	23.40%	7.120	81.87%
Weibull	4Yr	8.159	-12.77%	4.035	109.61%
	5Yr	-1.012	-393.39%	3.699	489.46%
	6Yr	-0.923	-446.07%	4.647	263.00%
	7Yr	0.325	-276.73%	2.948	516.22%

Table 3

The following table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: $Market\ Price = a + b (Model\ Price) + error$. The option prices were calculated under the assumption of the **g-and-h** distribution at the expiry of the option.

	Rate	a	b	Rsquare	Std. Error
2Yr	5%	-0.0005	0.9933	0.99997	0.0670
	5.50%	0.0100	1.0158	0.99871	0.1983
	6%	0.0241	1.0046	0.99845	0.1125
	6.50%	-0.0007	0.9995	0.99959	0.0316
	7%	-0.0315	0.8799	0.92324	0.1949
	7.50%	-0.0007	0.8941	0.97227	0.0490
3Yr	5%	-0.0001	0.9949	0.99999	0.0406
	5.50%	-0.0011	1.0013	0.99912	0.1641
	6%	0.0377	1.0138	0.99705	0.1554
	6.50%	-0.0013	1.0025	0.99918	0.0485
	7%	0.0091	0.8923	0.91629	0.1567
	7.50%	-0.0036	0.8745	0.95948	0.0631
4Yr	5%	0.0000	0.9954	0.99998	0.0261
	5.50%	-0.0020	0.9973	0.99980	0.0974
	6%	0.0264	1.0124	0.99738	0.1426
	6.50%	-0.0006	1.0122	0.99924	0.0474
	7%	-0.0295	0.9635	0.95649	0.0948
	7.50%	-0.0023	0.8988	0.98323	0.0419
5Yr	5.50%	-0.0002	0.9976	0.99997	0.0417
	6%	0.0069	1.0052	0.99851	0.1055
	6.50%	0.0003	1.0093	0.99971	0.0293
	7%	0.0714	0.9376	0.95701	0.0767
	7.50%	-0.0012	0.9644	0.99741	0.0168
	8%	-0.0002	0.8609	0.82012	0.0472
6Yr	5.50%	0.0000	0.9980	0.99998	0.0277
	6%	0.0002	1.0031	0.99935	0.0707
	6.50%	0.0003	1.0089	0.99988	0.0187
	7%	0.0836	0.9358	0.97203	0.0569
	7.50%	-0.0009	0.9755	0.99847	0.0130
	8%	0.0035	0.8800	0.88630	0.0337
7Yr	5.50%	0.0000	0.9994	1.00000	0.0078
	6%	-0.0005	1.0010	0.99992	0.0258
	6.50%	-0.0002	1.0085	0.99990	0.0172
	7%	0.0607	0.9533	0.97822	0.0456
	7.50%	-0.0008	0.9846	0.99870	0.0122
	8%	-0.0151	0.9881	0.95997	0.0189

Table 4

The following table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: $Market\ Price = a + b (Model\ Price) + error$. The option prices were calculated under the assumption of the **GB2** distribution at the expiry of the option.

Tenor	Rate	a	b	Rsquare	Std. Error
2Yr	5%	-0.0011	0.9762	0.99998	0.0520
	5.50%	0.0747	1.0505	0.99085	0.5285
	6%	0.0253	1.0431	0.99635	0.1728
	6.50%	-0.0071	1.0328	0.99669	0.0893
	7%	0.1947	0.6251	0.79395	0.3193
	7.50%	0.0043	0.6691	0.87344	0.1046
3Yr	5%	-0.0004	0.9777	0.99999	0.0374
	5.50%	0.0883	1.0029	0.98872	0.5880
	6%	0.0595	1.0610	0.99332	0.2337
	6.50%	-0.0048	1.0308	0.99863	0.0625
	7%	0.3483	0.6246	0.81655	0.2320
	7.50%	0.0004	0.6459	0.88828	0.1048
4Yr	5%	0.0000	0.9781	1.00000	0.0100
	5.50%	0.0085	0.9903	0.99779	0.3259
	6%	0.0579	1.0494	0.99034	0.2737
	6.50%	-0.0034	1.0476	0.99824	0.0723
	7%	0.4083	0.6498	0.82404	0.1906
	7.50%	0.0021	0.6850	0.91288	0.0955
5Yr	5.50%	-0.0003	0.9853	0.99998	0.0336
	6%	0.0445	1.0266	0.98795	0.3002
	6.50%	-0.0009	1.0552	0.99916	0.0499
	7%	0.5677	0.6275	0.75442	0.1832
	7.50%	0.0009	0.7938	0.97254	0.0547
	8%	0.0763	0.3232	0.57050	0.0729
6Yr	5.50%	-0.0002	0.9854	0.99998	0.0254
	6%	0.0256	1.0120	0.99161	0.2544
	6.50%	-0.0008	1.0574	0.99918	0.0495
	7%	0.5662	0.6499	0.78838	0.1564
	7.50%	-0.0033	0.8466	0.98917	0.0347
	8%	0.0871	0.3294	0.57349	0.0653
7Yr	5.50%	0.0000	0.9857	0.99998	0.0198
	6%	0.0106	1.0040	0.99595	0.1874
	6.50%	-0.0011	1.0576	0.99916	0.0504
	7%	0.4778	0.7104	0.83116	0.1269
	7.50%	-0.0032	0.8504	0.99204	0.0303
	8%	0.0655	0.3935	0.60675	0.0593

Table 5

The following table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: $Market\ Price = a + b (Model\ Price) + error$. The option prices were calculated under the assumption of the **Burr-3** distribution at the expiry of the option.

Tenor	Rate	a	b	Rsquare	Std. Error
2Yr	5%	0.6140	2.3450	0.9563	2.3746
	5.50%	1.8364	1.6358	0.8275	2.2954
	6%	1.6249	1.1249	0.6964	1.5758
	6.50%	0.2012	0.6965	0.8834	0.5302
	7%	0.9715	0.2915	0.1832	0.6358
	7.50%	0.0086	0.2276	0.8424	0.1167
3Yr	5%	0.1023	2.6729	0.9815	1.4858
	5.50%	1.7636	1.6540	0.8427	2.1964
	6%	1.4936	1.1707	0.7519	1.4244
	6.50%	0.2215	0.7392	0.8429	0.6695
	7%	1.3101	0.1867	0.1116	0.5106
	7.50%	0.0242	0.2247	0.7290	0.1633
4Yr	5%	0.0033	1040.0682	0.9975	0.2946
	5.50%	0.5670	727.1868	0.9444	1.6325
	6%	1.3738	487.2378	0.7962	1.2569
	6.50%	0.1112	318.4974	0.8229	0.7253
	7%	2.0491	-25.0283	0.0123	0.4517
	7.50%	0.0271	89.4768	0.6053	0.2032
5Yr	5.50%	0.3332	1.9056	0.9488	1.7394
	6%	1.3597	1.1479	0.7895	1.2550
	6.50%	0.1149	0.7694	0.8918	0.5671
	7%	1.3312	0.1560	0.1851	0.3337
	7.50%	0.0385	0.2451	0.7862	0.1527
	8%	0.1825	0.0456	0.0500	0.1084
6Yr	5.50%	0.1095	2.0540	0.9658	1.2119
	6%	1.2254	1.1535	0.8110	1.2071
	6.50%	0.0982	0.7744	0.9025	0.5389
	7%	1.2992	0.1438	0.1886	0.3063
	7.50%	0.0387	0.2494	0.7986	0.1497
	8%	0.1602	0.0475	0.0686	0.0965
	5.50%	0.0331	2.2310	0.9770	0.7238
7Yr	6%	0.9897	1.1761	0.8482	1.1472
	6.50%	0.0976	0.7798	0.9036	0.5394
	7%	1.2529	0.1349	0.2003	0.2761
	7.50%	0.0375	0.2565	0.8156	0.1456
	8%	0.1328	0.0518	0.0905	0.0902

Table 6

The following table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: $Market\ Price = a + b (Model\ Price) + error$. The option prices were calculated under the assumption of the **Lognormal** distribution at the expiry of the option.

Tenor	Rate	a	b	Rsquare	Std. Error
2Yr	5%	0.119954	2.716572	0.994050	0.875937
	5.50%	1.148203	1.805696	0.927341	1.489608
	6%	1.091202	1.060303	0.898276	0.912176
	6.50%	0.169850	0.826769	0.903803	0.481656
	7%	0.390392	0.272722	0.652974	0.414421
	7.50%	0.014723	0.170521	0.798610	0.131935
3Yr	5%	0.011543	2.728297	0.998902	0.361917
	5.50%	1.025577	1.837937	0.932937	1.433926
	6%	1.080353	1.081937	0.894069	0.930777
	6.50%	0.176353	0.863081	0.878630	0.588423
	7%	0.753561	0.208199	0.565883	0.356923
	7.50%	0.030101	0.165874	0.698128	0.172311
4Yr	5%	0.000084	2.732758	0.999966	0.034790
	5.50%	0.347975	1.927794	0.971758	1.164011
	6%	2.066477	0.961362	0.761322	1.360259
	6.50%	0.122611	0.957015	0.896325	0.554900
	7%	1.059690	0.153940	0.482502	0.326924
	7.50%	0.036871	0.187868	0.716351	0.172277
5Yr	5.50%	0.045914	1.945380	0.993612	0.614218
	6%	3.119459	0.879656	0.618824	1.688711
	6.50%	0.085347	1.170790	0.927678	0.463547
	7%	1.306191	0.117218	0.407656	0.284515
	7.50%	0.032117	0.254855	0.852192	0.127000
	8%	0.137309	0.033437	0.306676	0.092636
6Yr	5.50%	0.000386	1.933716	0.999950	0.046274
	6%	3.082462	0.997222	0.613880	1.725424
	6.50%	0.030777	1.357933	0.975705	0.269068
	7%	1.291102	0.121728	0.399696	0.263426
	7.50%	0.023082	0.299831	0.887799	0.111704
	8%	0.142690	0.031897	0.265634	0.085696
7Yr	5.50%	-0.000089	1.945277	0.999957	0.031342
	6%	2.163677	1.309089	0.717493	1.564987
	6.50%	0.030656	1.359038	0.975919	0.269535
	7%	1.263013	0.124614	0.345216	0.249856
	7.50%	0.022653	0.304669	0.896671	0.109017
	8%	0.143189	0.029764	0.174864	0.085872

Table 7

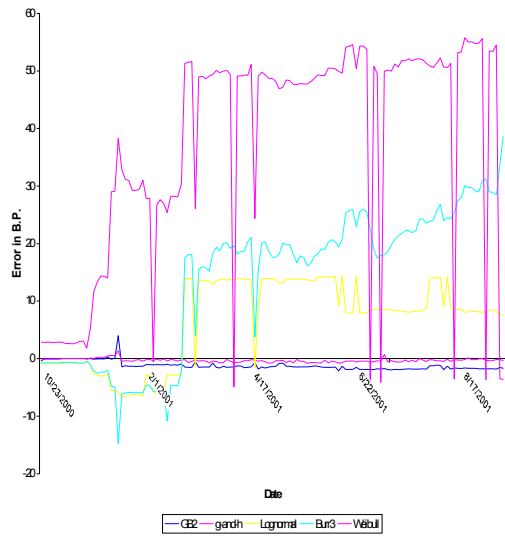
The following table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: $Market\ Price = a + b (Model\ Price) + error$. The option prices were calculated under the assumption of the **Weibull** distribution at the expiry of the option.

Tenor	Rate	a	b	Rsquare	Std. Error
2Yr	5%	15.83270	0.31192	0.03012	11.18338
	5.50%	12.77371	0.19399	0.01811	5.47594
	6%	7.54973	0.21843	0.02776	2.82003
	6.50%	0.94395	0.68671	0.01690	1.53977
	7%	1.83946	0.33560	0.11415	0.66213
3Yr	7.50%	0.15165	0.40360	0.00761	0.29288
	5%	5.88845	0.67063	0.11066	10.29938
	5.50%	12.82122	0.17787	0.01648	5.49133
	6%	7.68061	0.17811	0.01918	2.83223
	6.50%	1.02770	0.69606	0.01428	1.67692
4Yr	7%	1.86785	0.22706	0.08134	0.51922
	7.50%	0.17255	0.21415	0.00170	0.31335
	5%	0.94986	0.83925	0.36618	4.73121
	5.50%	10.72099	0.31356	0.03295	6.81132
	6%	7.51586	0.16473	0.02180	2.75378
5Yr	6.50%	0.91284	0.82139	0.05244	1.67757
	7%	1.81711	0.17795	0.07469	0.43716
	7.50%	0.17637	0.43832	0.02309	0.31971
	5.50%	6.49870	0.57911	0.06171	7.44411
	6%	7.41300	0.15286	0.01901	2.70910
6Yr	6.50%	0.88423	0.81379	0.05335	1.67708
	7%	1.79881	0.14717	0.06562	0.35734
	7.50%	0.18990	0.61768	0.03778	0.32403
	8%	0.25774	0.22216	0.10756	0.10510
	5.50%	0.70981	0.94465	0.80429	2.89851
7Yr	6%	3.15807	0.63132	0.57707	1.80581
	6.50%	-0.00048	1.03239	0.99982	0.02332
	7%	1.19802	0.33314	0.38971	0.26561
	7.50%	-0.00088	0.88978	0.98144	0.04543
	8%	0.10589	0.40260	0.52841	0.06867
7Yr	5.50%	-0.00005	0.99021	0.99999	0.01126
	6%	1.02162	0.88485	0.85495	1.12137
	6.50%	-0.00075	1.03329	0.99977	0.02622
	7%	0.94130	0.43498	0.44510	0.23001
	7.50%	-0.00033	0.89415	0.98260	0.04473
	8%	0.21010	0.05401	0.04229	0.09251

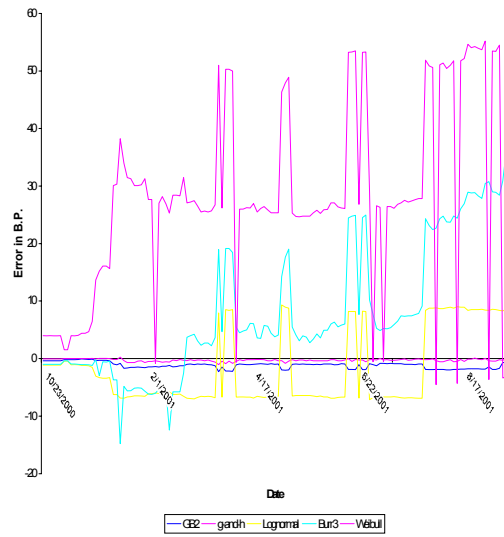
Table 8

The following table presents the summary of the correlation of errors (*market_price – model_price*) between the distributions we used to estimate the implied distribution. The time series of the error was computed by adding the errors for all the options of different strikes on a given day.

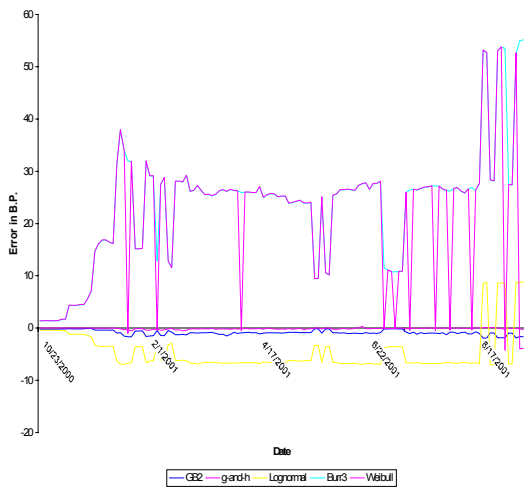
Tenor		GB2	g-and-h	Lognormal	Burr3	Weibull
2Yr	GB2	1	0.69	-0.55	-0.69	-0.54
	g-and-h		1	-0.56	-0.43	-0.38
	Lognormal			1.0	0.84	0.63
	Burr3				1	0.58
	Weibull					1
3Yr	GB2	1	0.59	-0.58	-0.60	-0.78
	g-and-h		1	-0.07	-0.06	-0.31
	Lognormal			1	0.84	0.47
	Burr3				1	0.52
	Weibull					1
4Yr	GB2	1	0.45	-0.10	-0.91	-0.62
	g-and-h		1	0.18	-0.29	-0.16
	Lognormal			1	0.19	-0.13
	Burr3				1	0.62
	Weibull					1
5Yr	GB2	1	0.59	0.86	-0.54	-0.76
	g-and-h		1	0.50	-0.12	-0.47
	Lognormal			1	-0.62	-0.76
	Burr3				1	0.36
	Weibull					1
6Yr	GB2	1	0.65	0.84	-0.57	-0.21
	g-and-h		1	0.56	-0.18	-0.16
	Lognormal			1	-0.59	-0.24
	Burr3				1	0.21
	Weibull					1
7Yr	GB2	1	0.45	0.70	-0.40	0.11
	g-and-h		1	0.45	-0.12	0.15
	Lognormal			1	-0.44	0.05
	Burr3				1	0.04
	Weibull					1



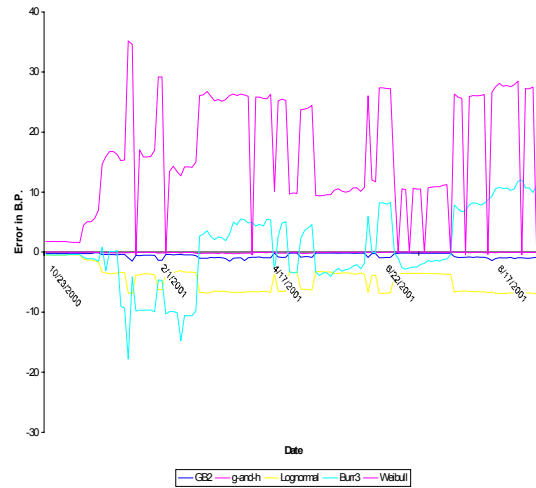
(a)



(b)



(c)



(d)

Figure 1: Model Errors – (a) 2-yr maturity, (b) 3-yr maturity, (c) 4-yr maturity, and (d) 5-yr maturity. The figures show the error between the model and market prices of the options under different distributional assumptions.

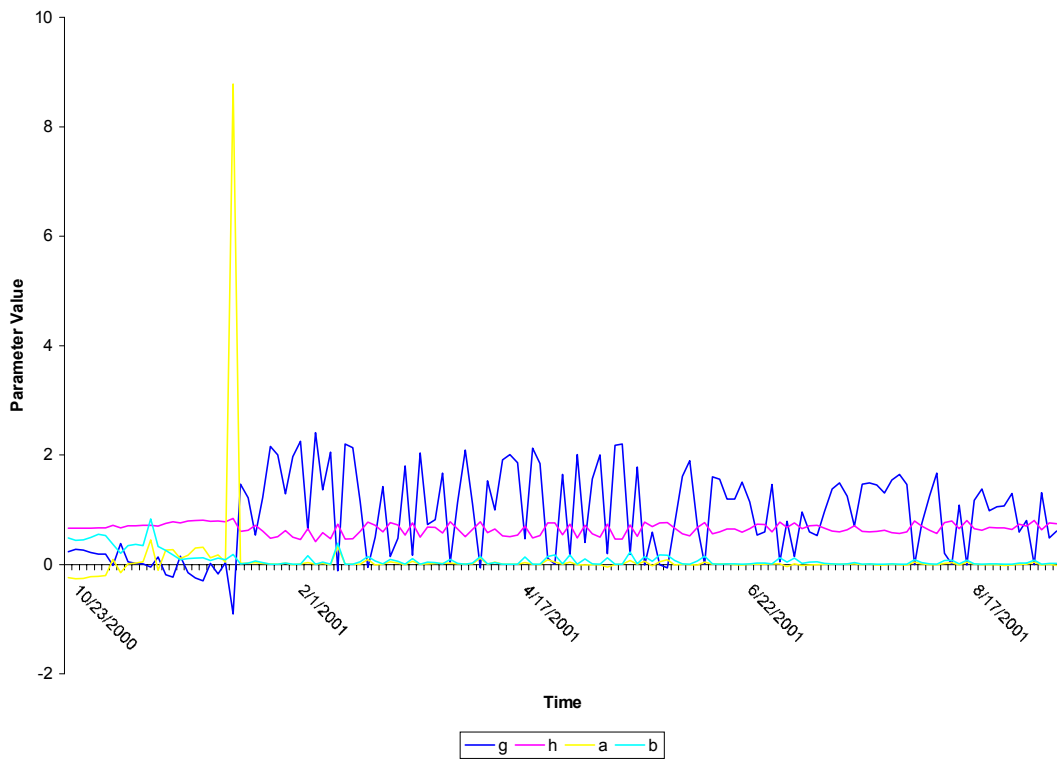


Figure 2: Implied Parameter Estimates for the g-and-h Distribution

The figure shows the average estimates of the parameter values across all maturities.

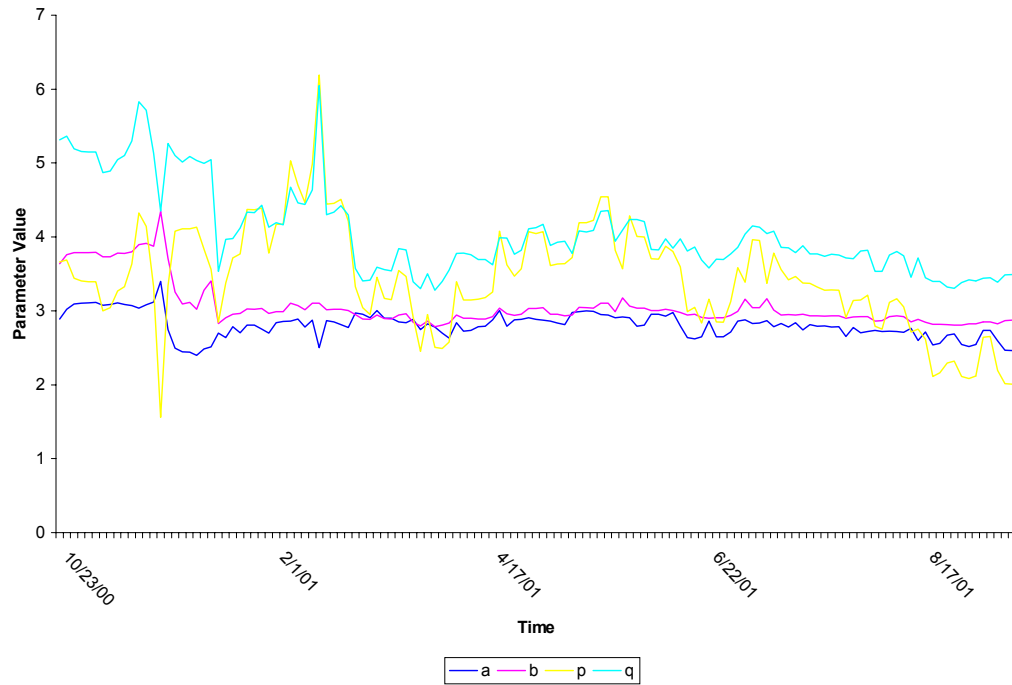


Figure 3: Implied Parameter Estimates for the GB2 Distribution

The figure shows the average estimates of the parameter values across all maturities.

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