Decomposing and Valuing the Callable Convertible Bonds:
A New Method Based on Exotic Options

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Abstract: In the framework of Black-Scholes-Merton option pricing models, by employing exotic options instead of plain options or warrants, this paper presents an equivalent decomposition method for the Callable Convertible Bonds (CCB). Furthermore, the analytic valuation formulae for CCB are obtained by using the analytic formulae for those simpler securities decomposed from CCB. This method is validated by comparing with Monte Carlo simulation. Besides, the effects of call clauses, coupon clauses, soft call condition clauses and dividend payment are analyzed respectively. These give a lot of new insights into the valuation and analysis of CCB and much help to replicate and hedge CCB.

Key words: Callable convertible bonds; Equivalent decomposition; Up-and-out calls; American binary calls; Derivative pricing

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1. Introduction

Convertible bonds have been playing a major role in the financing of companies because of their appealing hybrid feature that provides investors with both the downside protection of ordinary bonds and the upside return of equities. In practice, there are many kinds of the convertible bonds with diversified additional clauses, such as call clauses, put clauses, reset clauses, screw clauses and negative pledge clauses and so on. Although convertible bonds in the developed derivative markets, such as American derivative market, are generally very complex, those in the developing derivatives markets, such as Chinese derivative market, are relative simple. Anyway, callable convertible bonds are the most popular.

There are many literatures on the pricing of the callable convertible bonds. The Black-Scholes-Merton option pricing theory has become the definitive theoretic foundation for pricing the convertible bonds since the pioneer paper by Ingersoll (1977a). For the first time, he obtained the analytic formulae for the callable convertible bonds by employing the theoretically reasonable one-factor (i.e. firm value) no-arbitrage model. From then on, the theoretical equilibrium price of the callable convertible bond is defined as the one that offers no arbitrage opportunity to either the holders or the issuers, on the assumption that the issuers execute the optimal call policy that maximizes the common shareholder’s wealth (i.e. minimizes the value of this convertible bond) and that the holders execute the optimal conversion strategies that maximize the value of this convertible bond at each point in time.

The vast majority of subsequent theoretical research has focused on either
extending Ingersoll’s work to more complicated convertible bonds or further relaxing
his “ideal conditions”. The two-factor (i.e. firm value and interest rate) no-arbitrage
model was presented firstly by Brennan and Schwartz (1980) and then developed
further by Buchan (1997), Carayannopoulos (1996) and Lvov et al. (2004). Although
these models based on firm value are theoretically appealing, they are impractical
because they involve some unobservable parameters (notably, the volatility of the firm
value). The more practical one-factor (i.e. stock price) no-arbitrage model was given
for the first time by McConnell and Schwartz (1986). However, in order to capture
default risk of convertible bonds, their models had to adopt the credit spread approach
that would necessarily result into the theoretic inconsistence because the convertible
bond is a kind of hybrid derivatives consisting of a debt part that is subject to default
risk and an equity part that is not. This theoretic inconsistence was reduced greatly by
Goldman Sachs (1994) and Tsiveriotis and Fernandes (1998). Subsequently, the more
reasonable two-factor (i.e. stock price and interest rate) no-arbitrage model was
proposed firstly by Cheung and Nelken (1994) and developed further by introducing
more reasonable interest rate models (Ho and Pfeffer, 1996; Yigitbasioglu, 2001).
Recently, the reduced-form approach has been adopted to consider default risk of the
convertible bonds (Davis and Lischka, 1999; Takahashi et al. 2001; Ayache, Forsyth
and Vetzal, 2003; Yigitbasioglu and Alexander, 2004, Liao and Huang, 2006). To sum
up, with the development of these models, the pricing results have become more and
more reasonable and accurate, and the mean of prediction errors can be less than 5%
(Barone-Adesi, Bermudez and Hatgioannides, 2003).
However, solving these models generally has to adopt intricate numerical procedures that are very difficult to accept for investors, especially investors in developing derivative markets. Furthermore, these models only gave the final pricing results and could not provide the investors with enough help to deeply understand the value components of the callable convertible bonds, especially the effect of every kind of typical clauses, and to conveniently replicate and effectively hedge them. Obviously, those can be done easily, if we are able to completely decompose the callable convertible bonds into simpler securities, which are tradable in the actual market and whose analytic valuation formulae have already been obtained.

Since 1960s, researchers have attempted to completely decompose the convertible bonds into simpler securities. Baumol, Malkiel and Quandt (1966) proposed that a non-callable convertible bond could be regarded either as its corresponding ordinary bond (with the same principal and coupons and maturity) with a detachable call option struck at the value of this ordinary bond, or as stocks plus a put option struck at the value of this ordinary bond, which is greater. However, in light of later research, their conclusion is demonstrably incorrect except when all market participants are risk-neutral. Ingersoll (1977a), under his “ideal conditions”, proved that a non-callable convertible bond had the same value as its corresponding ordinary bond plus an attached call warrant, and then obtained its analytic valuation formula. Nyborg (1996) extended his decomposition by allowing the underlying stock to pay dividends and the capital structure to be more complex. However, both Ingersoll and Nyborg viewed the non-callable convertible bonds as contingent claims
on the firm value. This makes parameter estimation very difficult since not all of firm assets are tradable. Connolly (1998, chapter 8) viewed them as derivatives on the underlying stock price, and completely decomposed a non-callable convertible bond into its corresponding ordinary bond and European call warrants. His decomposition is relatively reasonable.

However, in the existing literatures, until now there is no method to completely decompose the callable convertible bonds into simpler securities trading in the actual market. To all appearances, one callable convertible bond can be completely decomposed into three simpler securities: one ordinary bond, one call option (i.e. the holders’ convertible option) and another call option (i.e. the issuers’ callable option). However, this is not the case because there are negligible interactions between the holders’ convertible option and the issuers’ callable option. That is to say, these two embedded options are dependent with each other. If one is exercised, the other disappears simultaneously. As a result, the difference between the value of this callable convertible bond and the value of the portfolio of these three securities cannot be ignored (Ingersoll, 1977a; Ho and Pfeffer, 1996). Ingersoll (1977a) proved that a callable convertible discount bond had the same value as its corresponding ordinary discount bond plus an attached stock call warrant minus an additional term representing the cost of giving the callable option to the issuers. However, his model is impractical because he viewed the callable convertible discount bond as contingent claims on the firm value. Ho and Pfeffer (1996) considered the callable convertible bonds as derivatives on the underlying stock price and presented that the value of a
callable convertible bond was equal to its investment value (i.e. the value of its corresponding ordinary bond) plus its embedded warrant value minus its forced conversion value. However, they only demonstrated the importance of its forced conversion value and did not obtain the analytic valuation formula for it.

In a word, none of these decomposition methods above is good enough to illustrate the value components of the callable convertible bonds and to conveniently replicate and effectively hedge them. As a matter of fact, due to the interactions between the embedded convertible option and the embedded callable option, the callable convertible bond is equivalent to its corresponding ordinary bond (with the same principal and coupons and maturity) embedded with a peculiar path-dependent exotic option, whose exercise price and exercise time are indeterminate and whose profit is capped. Thus, inevitably, if a callable convertible bond is decomposed with only non-path-dependent plain options or warrants, there must be some residuals (e.g. the additional term and the forced conversion value mentioned above), which are not tradable securities in the actual market.

In this paper, in the framework of Black-Scholes-Merton option pricing models, according as the risk-neutral valuation principle, by employing simple exotic options instead of plain options or warrants, an equivalent decomposition method is presented for the Callable Convertible Bonds (CCB). Based on this method, one callable convertible discount bond can be completely decomposed into its corresponding ordinary discount bond and three kinds of simple exotic options: regular American binary calls with an immediately-made fixed payment, regular up-and-out calls and
regular American binary calls with a fixed payment deferred until maturity. Similarly, one coupon-bearing callable convertible bond can be completely decomposed into its corresponding ordinary bond and five kinds of simple exotic options. Intuitively and exactly, this method shows us the value components of CCB. And obviously it is very helpful to replicate and effectively hedge CCB.

Furthermore, the analytic valuation formulae for CCB are given by making full use of the existing analytic valuation formulae for these simpler securities decomposed from CCB. At the same time, these analytic formulae for CCB are validated by comparing with Monte Carlo simulation. Without doubt, these formulae, can produce pricing results very soon and be used to conduct sensitivity analysis conveniently, because they need not to consume huge computational resources that are necessary for numerical procedures. Besides, they can be used to analyze the effects of call clauses, coupon clauses, soft call condition clauses and dividend payment respectively. These give a lot of new insights into the valuation and analysis of CCB.

The remainder of this paper is organized as follows. In the next section, the assumptions and the rationale needed in this paper are explicated in detail. In section 3, we present an equivalent decomposition method for CCB. In section 4, the analytic valuation formulae are given. Subsequently, section 5 validates these formulae by comparing with Monte Carlo simulation. In section 6, we further analyze in detail the effect of every kind of typical clauses and dividend payments respectively. Section 7 concludes the paper.
2. Valuation framework

2.1. Assumptions

(a) The framework of Black-Scholes-Merton option pricing models is adopted. It’s well-known that this framework is very rigorous and later has been relaxed gradually in order to price plain stock options more exactly. However, this framework has still often been adopted in order to obtain analytic valuation formulae for those complex derivative securities. As we know, in the Black-Scholes-Merton framework, capital market is perfect and efficient; the term structure of the risk-free rate of interest is flat; there is no riskless arbitrage opportunity; and the underlying stock price follows the diffusion process below.

\[ dS = \mu S d\tau + \sigma S dW \]  

where the variable \( W \) follows a standard Wiener process under probability measure \( Q \); \( \mu \) and \( \sigma \) are the expected rate of return and volatility of the underlying stock price respectively. Let \( r \) denote the continuous risk-free interest rate. Furthermore, \( \sigma \) and \( r \) are assumed here to be constant\(^\text{**}\). Dividend payments are not allowed for a while so that optimal conversion strategies can be obtained easily. In the section 6.6, we will discuss further the effect of dividend payments.

(b) All investors prefer more wealth to less. That is, the holders of the convertible bonds always seek to maximize their own wealth; the issuers of the convertible bonds,\(^\text{**}\)

\(^\text{**}\)Since Black and Scholes are only interested in the underlying asset price at maturity, they can allow \( \sigma \) and \( r \) to be known functions of time. However, CCB and exotic options involved in this paper depend in complex ways on the time paths of these variables. Simply, we assume here that these variables are constant through time.
as the deputies of the shareholders, act at all times to maximize the shareholders’ wealth, i.e. the underlying stock price.

(c) Both the holders and the issuers behave with symmetric market rationality. This implies that both the holders and the issuers are completely rational and one part can expect the optimal behaviors of the other. This is also the assumption by Ingersoll (1977a).

(d) The potential dilution, which results from the possible conversion in the future, is already reflected in the current underlying stock price. That is to say, the convertible bonds “can be valued without correction for dilution by using the volatility of the quoted share” (Connolly, 1998).

2.2. The rationale

According as the risk-neutral valuation principle, in the risk-neutral world, the expected return on all securities is the risk-free interest rate and the present value of any payoff can be obtained by discounting its expected value at the risk-free interest rate (Cox and Ross, 1976). Although the risk-neutral world is merely an artificial device for pricing derivative securities in the framework of the Black-Scholes-Merton option models, the valuation formulae obtained in the risk-neutral world are valid in all worlds. “When we move from a risk-neutral world to a risk-averse world, two things happen. The expected growth rate in the stock price changes and the discount rate that must be used for any payoff from the derivatives changes. It happens that these two changes always offset each other exactly (Hull, 2000, chapter 11).”
As seen in Harrison and Kreps (1979), in the risk-neutral world, the diffusion process that the underlying stock price follows becomes

\[ dS = rSdT + \sigma Sd\tilde{W} \]  

(2)

where the variable \( \tilde{W} \) follows another standard Wiener process under the risk-neutral probability measure \( \tilde{Q} \), which is equivalent to the probability measure \( Q \). In this way, the expected return rate becomes the risk-free interest rate, but the volatility has no change.

3. Decomposing the callable convertible bond

3.1. Definition

In this paper, we focus on the usual Callable Convertible Bond (CCB) whose conversion feature and call feature are defined as follows. More specifically, (d1) they entitle the holders to convert them into common shares at the predetermined conversion price at any time in the future; (d2) they entitle the issuers to call them back at the predetermined call price at any time in the future; (d3) they have no call notice period, i.e. when the issuers announce a call, the holders must immediately choose between accepting the call price in cash or converting them into the underlying stocks; (d4) both the conversion price and the call price are constant; (d5) they have the usual screw clauses, i.e. upon conversion the holders can not receive accrued interests any longer; (d6) they have no put clauses and reset clauses and other non-standard clauses. In the section 6.5, we will discuss further when they have the soft call condition clauses.
Although CCB with these clauses are relative simple, their value components are very similar with those of more complex convertible bonds with various flavor and forms. Therefore, if we are able to completely decompose these CCB into simpler securities trading in the actual market, we will better understand the value components of CCB and better replicate and hedge them, even the more complex convertible bonds.

Consider one CCB defined above. For convenience, we denote its face value by $B_F$, conversion price by $P$, call price by $B_c$, remaining time to maturity by $T$. Then, its conversion ratio, i.e. the number of shares of the underlying common stocks into which it can be converted, is $\left(\frac{B_F}{P}\right)$.

Without loss of generality, assume that it still has $N$ times payments of nominal coupons from now to maturity. Let $\tau_i (i = 1, \cdots, N)$ denote correspondingly the time span from now to the ex-coupon date. Obviously, $\tau_N = T$. Let $C_i (i = 1, \cdots, N)$ and $R_i (i = 1, \cdots, N)$ denote respectively the coupon amount and the coupon rate at time $\tau_i$. In this way, we can know that $C_i = B_F R_i$. Obviously, the coupon rate here is alterable. And let $Pv(T;C)$ denote the present value of all coming nominal coupons from now to maturity and $Fv(T;C)$ denote the future value of them at maturity. Let $Pv(\tau^*;C)$ denote the present value of all coming nominal coupons from now to the time $\tau^*$ at which the issuers will announce a call on their own initiative and $Fv(\tau^*;C)$ denote the future value of them at time $\tau^*$.

Besides, let $S_0$, $S_\tau$ and $S_T$ denote the underlying stock price respectively at current time zero, at any future time $\tau$ and at maturity $T$, where $0 < \tau \leq T$. Let
denote its theoretical value at current time zero and \( B(T; C) \) denote the theoretical value at current time zero of its corresponding ordinary bond (with the same principal and coupons and maturity), which is often referred to as the investment value.

### 3.2. Constraint Conditions

Based on the assumption (d) above, the conversion of CCB will not result in the immediate reduction of the underlying stock price because the underlying stock price has already reflected the potential dilution. Thus, its conversion value at any time \( \tau \) will be exactly equal to \( (B_{r}/P_{f})S_{r} \). Since CCB can be converted at any time in the future, its theoretical value must be at least as greater as its conversion value (McConnell and Schwartz, 1986). Otherwise, from arbitrage theory, riskless arbitrage profits can be obtained by purchasing CCB and exchanging for common stocks and selling immediately at the conversion value. In addition, the value of its embedded ordinary bond, i.e. its so-called investment value, can provide it with the downside protection at any time. Hence, its theoretical value at any time in the future before the call announcement and maturity must satisfy

\[
CCB(S_{\tau}, T - \tau; C) \geq \max \left[ B(T - \tau; C), (B_{r}/P_{f})S_{r} \right]
\]

(3)

Following McConnell and Schwartz (1986) and Barone-Adesi, Bermudez and Hatgjoannides (2003), due to the callable option, its theoretical value will not be allowed by the issuers to exceed the predetermined call price.

\[
CCB(S_{\tau}, T - \tau; C) \leq B_{c}
\]

(4)
Putting (3) and (4) together, we can obtain

$$\max \left[ B(T - \tau; C), (B_F/P) S_\tau \right] \leq CCB(S_\tau, T - \tau; C) \leq B_c$$  \hspace{1cm} (5)$$

If a call were to be announced at time $\tau^*$ prior to maturity, the holders would choose the more attractive of the two options: accepting the call price $B_c$ in cash or converting and obtaining the conversion value $(B_F/P) S_{\tau^*}$, where $S_{\tau^*}$ denote the underlying stock price at time $\tau^*$.

$$CCB\left(S_{\tau^*}, T - \tau^*; C\right) = \max\left[ (B_F/P) S_{\tau^*}, B_c \right] \text{ at call}$$  \hspace{1cm} (6)$$

If no call were to be announced prior to maturity, according to the optimal conversion strategies given in the next section, CCB would be held until maturity. At maturity, the holders can accept the balloon payment or convert and obtain the conversion value, which is greater. Due to the usual screw clauses, the balloon payment is $B_F + C_N$. Therefore, the final condition for CCB is

$$CCB(S_\tau, 0; C) = \max\left[ (B_F/P) S_\tau, B_F + C_N \right]$$  \hspace{1cm} (7)$$

3.3. Optimal conversion strategies

The holders of CCB defined above are entitled to convert one unit of CCB at any time in the future into $(B_F/P) \text{ units of shares of the underlying common stock.}$

Based on the assumption (b) above, optimal conversion strategies for the holders are those strategies that maximize the theoretical value of CCB.

**Theorem 1:** Given the assumptions in the section 2.1, it is optimal for the holders never to voluntarily convert the callable convertible bond defined in the section 3.1 except at maturity or the call announcement.
The proof of this theorem sees Appendix A. In fact, this theorem is similar with Ingersoll’s Theorem II (Ingersoll, 1977a) that a callable convertible security will never be exercised except at maturity or call. The only difference is that he viewed the callable convertible securities as the contingent claims on the firm value, but we view CCB as derivatives on the underlying stock price.

Prior to maturity, if a call were to be announced, from (6) the holders must choose immediately between accepting the call price in cash and converting CCB. Based on the assumption (c) above, the holders can expect the optimal call policy for the issuers. From Theorem 2 in the next section, it is optimal for the issuers to announce a call as soon as the underlying stock price reaches \( S^*_t = \left( \frac{B_r}{B_p} \right) P_t \), i.e. the conversion value reaches the call price, \( \left( \frac{B_r}{P_t} \right) S^*_t = B_r \). Therefore, upon the call announcement, the holders would be indifferent between accepting the call price in cash and converting.

If no call were to be announced prior to maturity, CCB would be held until maturity. At maturity, from the final condition (7), it is self-evident that the holders should voluntarily convert if the conversion value \( \left( \frac{B_r}{P_t} \right) S_T > B_r + C_N \), i.e. \( S_T > \left( 1 + \frac{C_N}{B_r} \right) P_t \), and should claim the balloon payment otherwise.

### 3.4. Optimal call policies

The issuers of CCB defined above are entitled to call CCB back at the predetermined call price at any time in the future. Based on the assumption (b),
optimal call policies for the issuers are those policies that maximize the underlying stock price or, what is the same thing, minimize the theoretical value of CCB.

**Theorem 2:** Given the assumptions in the section 2.1, it is optimal for the issuers to announce to call back the callable convertible bond defined in the section 3.1 as soon as the underlying stock price reaches

\[
S^*_t = \left( \frac{B_c}{B_r} \right) P_1.
\]

The proof of this theorem sees Appendix B. In fact, this theorem is similar with Ingersoll’s Theorem IV (Ingersoll, 1977a). Upon the call announcement, the holders will be in the same way indifferent between accepting the call price in cash and obtaining the conversion value, though he viewed CCB as the contingent claims on the firm value and we view CCB as derivatives on the stock price,.

In practice, however, the call policies by the issuers are not consistent with these theoretical works. The issuers generally delay announcing a call until the conversion value is substantially higher than the call price (Ingersoll, 1977b; Constantinides and Grundy, 1987). Some reasons are demonstrated by Jalan and Barone-Adesi (1995) and Ederington, Caton and Campbell (1997) and so on. In order to consider this inconsistency, following Barone-Adesi, Bermudez and Hatgioannides (2003), the restriction condition (4) can be modified by writing:

\[
CCB(S_T, T - \tau; C) \leq k B_c
\]  

where \( k \) is a conveniently chosen factor bigger than one. In the same way, we can obtain that it is optimal for the issuers to announce a call as soon as the underlying stock price reaches

\[
\hat{S}^*_t = k \left( \frac{B_c}{B_r} \right) P_1.
\]
3.5. The equivalent decomposition

Concerned with the ending of CCB defined in the section 3.1, based on the assumptions in the section 2.1 and the optimal conversion strategies in the section 3.3 and the optimal call policy in the section 3.4, there exist only three possible cases. For convenience, let \( P_2 = S^*_T = \left( \frac{B_c}{B_r} \right) P_1. \)

In the first case, the underlying stock price will reach \( P_2 \) prior to maturity, then the issuers will announce a call at once on their own initiative. At that time, the holders will be indifferent between accepting the call price in cash and converting. In the second case, the underlying stock price will not reach \( P_2 \) prior to maturity but at maturity will exceed the conversion price, and then CCB will be voluntarily converted at maturity by the holders on their own initiative. In the third case, the underlying stock price will neither reach \( P_2 \) prior to maturity nor at maturity exceed the adjusted conversion price \( \left( 1 + \frac{C_F}{P_C} \right) P_1 \), and then CCB will be redeemed at maturity by the issuers.

As a matter of fact, since the critical stock price \( P_2 \) can be regarded as the barrier of regular American binary calls with an immediately-made fixed payment, the payoff to CCB in the first case is similar with the payoffs to regular American binary calls. Furthermore, since the critical stock price \( P_2 \) and the adjusted conversion price \( \left( 1 + \frac{C_F}{P_C} \right) P_1 \) can be regarded respectively as the barrier and the exercise price of regular up-and-out calls, the payoff to CCB in the second case is similar with the payoffs to regular up-and-out calls. Therefore, firstly we can separate an American binary call and a regular up-and-out call from CCB respectively. Finally, CCB can be completely
decomposed into its corresponding ordinary bond and five kinds of simple exotic options through four steps as follows.

At the first step, off CCB, we strip \( \left( \frac{B_F}{P_1} \right) \) units of long regular American binary calls, denoted as \( ABC^u(S_0, T; P_2 - P_1, P_2) \), whose fixed payment \( (P_2 - P_1) \) is made immediately when the underlying stock price reaches for the first time the barrier \( P_2 \).

At the second step, from the rest, we separate \( \left( \frac{B_F}{P_1} \right) \) units of long regular up-and-out calls, denoted as \( UOC(S_0, T; \left( 1 + \frac{c_s}{\delta_r} \right) P_1, P_2) \), whose barrier is also \( P_2 \) and whose exercise price is the adjusted conversion price \( \left( 1 + \frac{c_s}{\delta_r} \right) P_1 \) because of the usual screw clauses.

After two steps above, the residual can be completely decomposed into three simpler securities. One is a short non-regular American binary call with a time-varying payment \( B_F + Fv(T; C) \) deferred until maturity when the underlying stock price reaches for the first time the barrier \( P_2 \), denoted as \( ABC^d(S_0, T; B_F + Fv(T; C), P_2) \). Another is a long non-regular American binary call with an immediately-made indeterminate payment \( B_F + Fv(\tau^*; C) \) when the underlying stock price reaches for the first time the barrier \( P_2 \), denoted as \( ABC^d\left(S_0, T; B_F + Fv(\tau^*; C), P_2\right) \). And the third one is its corresponding ordinary bond \( B(T; C) \).

In order to demonstrate better the value components of CCB, we continue the fourth step. In brief, \( ABC^d\left(S_0, T; B_F + Fv(T; C), P_2\right) \) can be further completely decomposed into one regular American binary call with a fixed payment \( B_F \).
deferred until maturity, denoted as $ABC^d(S_0,T;B_F,P_2)$, and one non-regular American binary call with a time-varying payment $Fv(T;C)$ deferred until maturity, denoted as $ABC^d(S_0,T;Fv(T;C),P_2)$. $ABC^i(S_0,T;B_F + Fv(\tau^*;C),P_2)$ can be further completely decomposed into one regular American binary call with an immediately-made fixed payment $B_F$, denoted as $ABC^i(S_0,T;B_F,P_2)$, and one non-regular American binary call with an immediately-made indeterminate payment $Fv(\tau^*;C)$, denoted as $ABC^i(S_0,T;Fv(\tau^*;C),P_2)$.

**Theorem 3:** Given the assumptions in the section 2.1, one callable convertible bond defined in the section 3.1 has the same value at any time as the portfolio consisting of $(B_F/P_1)$ units of long regular American binary calls $ABC^i(S_0,T;P_2 - P_1,P_2)$, $(B_F/P_1)$ units of long regular up-and-out calls $UOC(S_0,T;\left(1 + \frac{C_{id}}{\delta_F}\right)P_1,P_2)$, one short regular American binary call $ABC^d(S_0,T;B_F,P_2)$, one short non-regular American binary call $ABC^d(S_0,T;Fv(T;C),P_2)$, one long regular American binary call $ABC^i(S_0,T;B_F,P_2)$, one long non-regular American binary call $ABC^i(S_0,T;Fv(\tau^*;C),P_2)$, and its corresponding ordinary bond $B(T;C)$. This can be shown as the following equation.

\[
\begin{align*}
CCB(S_0,T;C) &= (B_F/P_1)ABC^i(S_0,T;P_2 - P_1,P_2) + (B_F/P_1)UOC(S_0,T;\left(1 + \frac{C_{id}}{\delta_F}\right)P_1,P_2) \\
&\quad + ABC^i(S_0,T;B_F,P_2) - ABC^d(S_0,T;B_F,P_2) \\
&\quad + ABC^i(S_0,T;Fv(\tau^*;C),P_2) - ABC^d(S_0,T;Fv(T;C),P_2) \\
&\quad + B(T;C)
\end{align*}
\]
The proof of this theorem is proved in Appendix C. Obviously, the equation (9) demonstrates fully the value components of CCB. It’s worth noting that 

\[ ABC^d (S_0, T; Fv(T; C), P_2) \] and \[ ABC^u (S_0, T; Fv(\tau^*; C)) \] are non-regular American binary calls. Fortunately, both of them result only from coupon payments and the holders take the short position in the former and the long position in the latter. It turns out that their total contribution to the value of CCB is relatively small, especially at near maturity and low current stock price.

In fact, \[ ABC^u (S_0, T; B_f P_2) \] and \( (B_f / P_1) \) units of \( ABC^u (S_0, T; P_2 - P_1, P_2) \) may be merged into \( ABC^u (S_0, T; (B_f P_2 / P_1), P_2) \), whose fixed payment \( (B_f P_2 / P_1) \) is made immediately. Then, the equation (9) becomes

\[
CCB(S_0, T; C) = ABC^u (S_0, T; (B_f P_2 / P_1), P_2) + (B_f / P_1) UOC \left( S_0, T; \left(1 + \frac{C_i}{\pi_f}\right) P_1, P_2 \right)
\]

\[
- ABC^d (S_0, T; B_f, P_2) + ABC^u (S_0, T; Fv(\tau^*; C), P_2)
- ABC^d (S_0, T; Fv(T; C), P_2) + B(T; C)
\]

This equation implies that CCB can be completely replicated with only five kinds of exotic option and its corresponding ordinary bond.

Let \( C_i = 0 (i = 1, \ldots, N) \), then CCB retrogresses to the callable convertible discount bond. Accordingly, the equation (10) becomes

\[
CCB(S_0, T; 0) = ABC^d (S_0, T; (B_f P_2 / P_1), P_2) + (B_f / P_1) UOC (S_0, T; P_1, P_2) - ABC^d (S_0, T; B_f, P_2) + B(T; 0)
\]

This equation implies that the callable convertible discount bond can be completely replicated with only three regular exotic options and its corresponding ordinary discount bond.
Let \( B_c \to +\infty \), then \( P_2 \to +\infty \), the callable option will never be exercised. Then, CCB retrogresses to the non-callable convertible bond. Accordingly, the equation (10) becomes

\[
CB(S_0, T; C) = \left( \frac{B_F}{P_1} \right) W \left( S_0, T; \left( 1 + \frac{\mu}{\sigma^2} \right) P_1 \right) + B(T, C)
\]

where \( W \left( S_0, T; \left( 1 + \frac{\mu}{\sigma^2} \right) P_1 \right) \) denotes a European call warrant with the exercise price \( \left( 1 + \frac{\mu}{\sigma^2} \right) P_1 \) and the remaining time to maturity \( T \). This equation implies that the non-callable convertible bonds can be completely replicated with European call warrants and its corresponding ordinary bond. In essence, this equation is the same as the one derived from the binomial tree method by Connolly (1998, chapter 8).

4. The analytic valuation formula

For regular American binary calls and regular up-and-out calls mentioned above, the analytic formulae have already been obtained in the Black-Scholes-Merton framework by Rubinstein and Reiner (1991a and 1991b). For the non-regular American binary call \( ABC^i(S_0, T; F_v(T^*; C), P_2) \), the analytic formula has been derived in Appendix D. In short, the analytic formulae for these securities decomposed from CCB can be directly expressed below.

\[
ABC^i (S_0, T; P_2 - P_1, P_2) = (P_2 - R) \left[ \left( P_2 / S_0 \right)^{\beta / \sigma^2} N(-\alpha_1) + \left( P_2 / S_0 \right)^{(\beta - \mu) / \sigma^2} N(-\alpha_2) \right]
\] (13)
\[ UOC \left( S_0, T; \left(1 + \frac{C_i}{N^2} \right) P_1, P_2 \right) \]
\[ = \left[ S_0 N \left( d_1 \right) - \left(1 + \frac{C_i}{N^2} \right) P e^{-\mu T} N \left( d_1 - \sigma \sqrt{T} \right) \right] \]
\[ - \left[ S_0 N \left( d_2 \right) - \left(1 + \frac{C_i}{N^2} \right) P e^{-\mu T} N \left( d_2 - \sigma \sqrt{T} \right) \right] + \left[ S_0 \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_3 \right) - \left(1 + \frac{C_i}{N^2} \right) P e^{-\mu T} \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_3 + \sigma \sqrt{T} \right) \right] \]
\[ - \left[ S_0 \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_4 \right) - \left(1 + \frac{C_i}{N^2} \right) P e^{-\mu T} \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_4 + \sigma \sqrt{T} \right) \right] \]
\[ \left( 14 \right) \]

\[ ABC^d \left( S_0, T; B_F, P_2 \right) = B_F e^{-\mu T} \left[ \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -a_1 \right) + N \left( -a_4 \right) \right] \]
\[ \left( 15 \right) \]

\[ ABC^i \left( S_0, T; B_F, P_2 \right) = B_F \left[ \left( P_2 / S_0 \right)^{\left(\mu + \hat{\mu}\right) / \sigma^2} N \left( -a_1 \right) + \left( P_2 / S_0 \right)^{\left(\mu - \hat{\mu}\right) / \sigma^2} N \left( -a_2 \right) \right] \]
\[ \left( 16 \right) \]

\[ ABC^d \left( S_0, T; FV \left( T; C \right), P_2 \right) = P_F \left[ \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -a_3 \right) + N \left( -a_4 \right) \right] \]
\[ \left( 17 \right) \]

\[ ABC^i \left( S_0, T; FV \left( \tau^*; C \right), P_2 \right) \]
\[ \left[ \sum_{i=1}^{N-1} B_F R_i e^{-\tau_i} \left[ \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -a_3 \right) + N \left( -a_4 \right) \right] \right] \]
\[ \left( 18 \right) \]

\[ B(T; C) = B_F e^{-\mu T} + \sum_{i=1}^{N} C_i e^{-\tau_i} \]
\[ \left( 19 \right) \]

where, \( \bar{\mu} = r - \frac{1}{2} \sigma^2 \), \( \mu = r + \frac{1}{2} \sigma^2 \), \( \hat{\mu} = \left( \bar{\mu}^2 + 2r \sigma^2 \right)^{1/2} \), \( a_1 = \left[ \ln \left( P_2 / S_0 \right) + \bar{\mu} T \right] / \left( \sigma \sqrt{T} \right) \), \( a_2 = \left[ \ln \left( P_2 / S_0 \right) - \bar{\mu} T \right] / \left( \sigma \sqrt{T} \right) \), \( d_1 = \left[ \ln \left( S_0 \left( 1 + C_N / B_F \right) P_1 \right) + \mu T \right] / \left( \sigma \sqrt{T} \right) \), \( d_2 = \left[ \ln \left( S_0 / P_2 \right) + \bar{\mu} T \right] / \left( \sigma \sqrt{T} \right) \), \( d_3 = \left[ \ln \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_3 \right) - \left(1 + \frac{C_i}{N^2} \right) P e^{-\mu T} \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_3 + \sigma \sqrt{T} \right) \right] \), \( d_4 = \left[ \ln \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_4 \right) - \left(1 + \frac{C_i}{N^2} \right) P e^{-\mu T} \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_4 + \sigma \sqrt{T} \right) \right] \), \( d_5 = \left[ \ln \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_4 \right) - \left(1 + \frac{C_i}{N^2} \right) P e^{-\mu T} \left( P_2 / S_0 \right)^{2\mu / \sigma^2} N \left( -d_4 + \sigma \sqrt{T} \right) \right] \), \( a_3 = \left[ \ln \left( P_2 / S_0 \right) + \bar{\mu} T \right] / \left( \sigma \sqrt{T} \right) \), \( a_4 = \left[ \ln \left( P_2 / S_0 \right) - \bar{\mu} T \right] / \left( \sigma \sqrt{T} \right) \), \( a_5 = \left[ \ln \left( P_2 / S_0 \right) + \bar{\mu} \tau_i \right] / \left( \sigma \sqrt{\tau_i} \right) \), \( a_6 = \left[ \ln \left( P_2 / S_0 \right) - \bar{\mu} \tau_i \right] / \left( \sigma \sqrt{\tau_i} \right) \) and \( N(x) \) is the cumulative probability distribution function for a variable \( x \) that is normally distributed with a mean of zero and a standard deviation of 1.0.

By substituting the equations (13) through (19) into the equation (10), the analytic formulae can be obtained easily for CCB defined in the section 3.1. Despite
the seemingly complex form, this formula is theoretically rigorous. Moreover, its derivation requires the same preconditions about capital markets as the Black-Scholes option pricing formulae. Besides, to get the pricing result, we only need to estimate \( \sigma \), just like the Black-Scholes option pricing formulae, too.

Besides, widespread use of this formula in practice could be expected owing to its several obvious advantages below. First, it can be used to quickly estimate the value of CCB without consuming huge computation resource always required by numerical procedures. Second, base on it, the important Greeks (such as delta, theta and gamma) for risk management can be directly calculated. Third, it may be used for sensitivity analysis that can give much help to design CCB. Four, it may also help investors seize possible riskless arbitrage opportunities between CCB and its duplicate portfolio mentioned in Theorem 3.

5. Validity Test

To assess the validity of the equivalent decomposition above, we have compared the pricing results from our analytic formulae with those from Monte Carlo simulation (Boyle, Broadie and Glasserman 1997). We adopt the pricing results from Monte Carlo simulation as benchmarks because Monte Carlo simulation has been widely considered as an essential tool in the pricing of daily monitored derivative securities. In this paper, Monte Carlo prices are computed by using 10,000 simulation paths on assumption that there are 252 closing prices per year, i.e. \( \Delta t = 1/252 \). Moreover, the antithetic variable technique for variance reduction is adopted.
Since our analytic valuation formulae for CCB are obtained in the continuous context, its pricing results for the daily monitored CCB consequentially have continuity errors. In order to remove continuity errors, we have adopted the continuity correction by Broadie, Glasserman and Kou (1997). Specifically, the original barrier $P_2$ is adjusted to be $P_2 \exp\left(\beta \sigma \sqrt{\Delta t}\right)$, where $\beta \approx 0.5826$.

Without loss of generality, consider a numerical example of the daily monitored CCB as follows. $B_F = $1000, $R_i = 0.04 \left(i = 1, \cdots, N\right)$, $P_i = $100, $B_c = $120, $r = 0.03$, $\sigma = 0.3$. Since both the current underlying stock price and the remaining time to maturity are state variables, comparisons are made in two different cases. In the first case, we set the remaining time to maturity to be five years and the current stock price to be variable within the range from $30$ to $120$, which is equally divided into 50 intervals, i.e. \(\Delta S = (120 - 30)/50 = $1.8\). In the second case, we set the current stock price to be $100$ and the remaining time to maturity to be variable within the range from zero to five years, which is equally divided into 50 intervals, i.e. \(\Delta \tau = 5/50 = 0.1\).

As illustrated in Fig. 1 (the current stock price is variable) and Fig. 2 (the remaining time to maturity is variable), the pricing results from our analytic formulae with the continuity correction (denoted as “Solution with correction”) are extremely close to those from Monte Carlo simulation (denoted as “Simulation”). The mean of percentage errors relative to the results from simulation is only 0.03% and the largest one does not exceed 0.08% in magnitude. Moreover, with the number of simulation paths increasing, the percentage errors become smaller. Hence, our analytic formulae
are indeed valid.

Fig. 1 Comparison when the current stock price is variable

Fig. 2 Comparison when the remaining time to maturity is variable

To illuminate the effect of continuity errors, comparison is also made between the pricing results from our analytic formulae with the continuity correction and ones without the continuity correction (denoted as “Solution without correction”), as
illustrated in Fig. 1 and Fig. 2. It can be concluded that the uncorrected results are always greater than the corrected ones. Moreover, the closer the current stock price is to the barrier \( P^2 \), the larger their differences are. The mean of percentage errors between them is 0.16\% and the largest one reaches 0.38\%. Hence, it is better to adopt continuity correction when our analytic formulae are applied to the discretely monitored CCB.

6. Analyzing the callable convertible bond

6.1. Theoretical value of CCB and state variables

On the assumptions stated in the section 2.1, the theoretical value of CCB depends on two state variables: its remaining time to maturity and the current underlying stock price. By employing the same numerical example in the section 5, its three-dimensional graph (see Fig. 3) has been plotted to demonstrate the relationships between its value and state variables. Here, its remaining time to maturity is set to be variable within the range from 0 to 5 years, which is equally divided into 50 intervals, i.e. \( \Delta \tau = \frac{5}{50} = 0.1 \); the current underlying stock price is set to be variable within the range from $30 to $120, which is also equally divided into 50 intervals, i.e. \( \Delta P = \frac{(120 - 30)}{50} = $1.8 \).

Fig. 3 shows clearly that its theoretical value is always an increasing function of the current underlying stock price. Fig. 3 also clearly shows that its theoretical value always rupture downside shortly after the ex-coupon dates and always increases gradually with the remaining time to maturity decreasing during the periods between
the coupon dates except the last period.

6.2. Value components of CCB and state variables

From the equation (9), the theoretical value of CCB consists of seven components. For convenience, denote 

\[ (B_F/P_i)ABC^i(S_0,T;P_2-P_1,P_2), \]

\[ ABC^i(S_0,T;B_F,P_2), \]

\[ ABC^i(S_0,T;Fv(T;C),P_2), \]

\[ ABC^d(S_0,T;B_F,P_2), \]

\[ ABC^d(S_0,T;Fv(T;C),P_2) \]

and \( (B_F/P_i)UOC(S_0,T;(1+C_C/P_2)P_1,P_2) \) by \( ABC^i0, ABC^i1, ABC^i2, ABC^d1, ABC^d2 \) and \( UOC \) respectively.

In order to display the contribution of each of these seven components to the whole under the different states, the same numerical example in the section 6.1 is also employed to plot the three-dimensional graphs of the relationships between each component and two state variables, illustrated as Fig. 4 through 10. Intuitively, Fig. 3 can be exactly obtained by overlaying Fig. 4 through 10 together in terms of the equation (9).
Fig. 4 Relationships between $B(T;C)$ and state variables

Fig. 5 Relationships between $ABCi0$ and state variables
Fig. 6 Relationships between $ABC_i1$ and state variables

Fig. 7 Relationships between $ABC_i2$ and state variables
Fig. 8 Relationships between $ABCd_1$ and state variables

Fig. 9 Relationships between $ABCd_2$ and state variables
6.3. The effect of coupon clauses

Without doubt, coupon payments must add the theoretical value of CCB. However, the added value by coupon payments must be less than the present value of all coming nominal coupons. There are two reasons. First, if CCB were to be called back prior to maturity, the nominal coupons hereafter would not be paid any longer. Second, if it were to be voluntarily converted at maturity, due to the screw clauses, the last nominal coupon would not be paid. Obviously, the added value by coupon payments should be the difference between $CCB(S_0, T; C)$ and $CCB(S_0, T; 0)$. In terms of the equations (9) and (11), the added value by coupon payments can be derived below.

$$CCBCoupon(S_0, T) = CCB(S_0, T; C) - CCB(S_0, T; 0)$$

$$= P_V(T; C) - \left\{ ABC^d \left( S_0, T; F_V(T; C), P_2 \right) - ABC^d \left( S_0, T; F_V(r^*; C), P_2 \right) \right\}$$

$$- \left( \frac{B_F}{P_1} \right) \left\{ UOC(S_0, T; P_1, P_2) - UOC \left( S_0, T; \left( 1 + \frac{C_0}{T} \right) P_1, P_2 \right) \right\}$$
By employing the same numerical example in the section 6.1, we also plot the three-dimensional graph of the relationships between the added value and state variables (see Fig. 11). Fig. 11 shows clearly that $CCBCoupon$ decreases with the current stock price increasing. Moreover, the curves of the relationship between $CCBCoupon$ and the remaining time to maturity look saw-toothed.

![Fig. 11 Relationships between $CCBCoupon(S_0, T)$ and state variables](image)

To further demonstrate the effect of coupon clauses, we have designed another indicator that is the ratio of the added value by coupon payment to the present value of all coming nominal coupons. It can be expressed below.

$$ Ratio(S_0, T) = \frac{CCBCoupon(S_0, T)}{Pv(T; C)} $$

Similarly, we plot its three-dimensional graph (see Fig. 12). Fig. 12 clearly shows that it decreases from 1 to 0 with the current stock price increasing. This is because the higher the current stock price is, the more possible it is for the issuers to call CCB back prior to maturity. In addition, it increases gradually with the remaining
time to maturity decreasing during the periods between the coupon dates except the last period, but ruptures downside shortly after the coupon dates, especially when CCB is near at-the-money.

Fig. 12 Relationships between $\text{Ratio}(S_0, T)$ and state variables

6.4. The effect of call clauses

Since the only difference between CCB and its corresponding non-callable convertible bond rests with call clauses, the effect of call clauses on the value of CCB can be obtained by subtracting the value of the former from the value of the latter. In terms of the equations (10) and (12), the analytic formulae for the effect of call clauses on the value of CCB can be derived below.
\[
Call(S_0,T;C) = CB(S_0,T;C) - CCB(S_0,T;C)
\]
\[
= (B_F/P_1)W(S_0,T;\left(1 + \frac{C_F}{B_F}\right)P_1) - ABC^{d}(S_0,T;\left(B_F/P_2\right),P_2)
- (B_F/P_1)UOC(S_0,T;\left(1 + \frac{C_F}{B_F}\right)P_1,P_2) + ABC^{d}(S_0,T;B_F,P_2)
- ABC^{d}(S_0,T;Fv(C,\tau^*),P_2) + ABC^{d}(S_0,T;Fv(C,T),P_2)
\] (22)

Its three-dimensional graph has also been plot (see Fig. 13) by employing the same numerical example in the section 6.1. Fig. 13 clearly shows that the effect of call clauses always increase with the current stock price and/or the remaining time to maturity.

Fig. 13 Relationships between \(CCBCall(S_0,T)\) and state variables

6.5. The effect of soft call condition clauses

Commonly, CCB are issued with soft call condition clauses that restrict the issuers to exercise the callable option. In this section, we analyze the effect of the soft call condition clauses where the issuers may call CCB back only if the underlying stock trades for at least less than a predetermined trigger price. Denote this trigger
price by $\bar{P}_2$. It must be greater than the critical stock price $P_2 = S^*_T = (B_e / B_P)P_1$, i.e.

$$(B_e / P_1)\bar{P}_2 > (B_e / P_1)P_2 = B_e,$$

or else from Theorem 2 the issuers will not be restricted by the soft call condition clauses to exercise the callable option. Obviously, the soft call condition clauses benefit the holders.

Based on the analysis in the section 3.3 and 3.4, since $\bar{P}_2 > P_2 = S^*_T$, it is optimal for the issuers to announce a call immediately as soon as the underlying stock price reaches the trigger price; then the holders must choose converting at once because at that time $$(B_e / P_1)\bar{P}_2 > (B_e / P_1)P_2 = B_e.$$ Except at the call announcement, the soft call condition clauses have no effect on the conversion optimal strategies in the section 3.3. Therefore, the analytic formulae for CCB with the soft call condition clauses can be easily derived with the same proof as the equation (10) by substituting the trigger price $\bar{P}_2$ for the critical stock price $P_2$.

$$CCB(S_0,T;C) = ABC^d(S_0,T;B_e\bar{P}_2/P_1,\bar{P}_2) + (B_e / P_1)UOC(S_0,T;1+\frac{C_2}{C_1})P_1,\bar{P}_2$$

$$\quad - ABC^d(S_0,T;B_e,\bar{P}_2) + ABC^d(S_0,T;Fv(T^*,C),\bar{P}_2)$$

$$\quad - ABC^d(S_0,T;Fv(T,C),\bar{P}_2) + B(T;C)$$

(23)

In this way, the analytic formula for the effect of the soft call condition clauses on the value of CCB can be derived below.

$$CCB_{Soft}(S_0,T;P_2,\bar{P}_2) = CCB(S_0,T;\bar{P}_2) - CCB(S_0,T;P_2)$$

(24)

Its three-dimensional graph has also been plot (see Fig. 14) with the same numerical example in the section 6.1 ($\bar{P}_2 = \$130$). Fig. 14 clearly shows that the effect of the soft call condition clauses always increases with the current stock price and/or the remaining time to maturity.
6.6. The effect of dividend payments

The results produced so far have assumed that the underlying stock pays no dividends. In practice, this is not usually true. In American stock market, dividends generally are paid quarterly and the convertible bonds are generally not dividend protected. In this section, we take account of the effect of dividend payments on the value of CCB. Assume that all $M$ times dividend payments during the remaining time to maturity of CCB and their ex-dividend dates in the future are anticipated. And let $\tau_1^d, \tau_2^d, \ldots, \tau_M^d$ denote moments immediately after going ex-dividend with $\tau_1^d < \tau_2^d < \ldots < \tau_M^d < T$. The dividends and the underlying stock prices corresponding to these times will be denoted by $D_1, D_2, \ldots, D_M$ and $S_1, S_2, \ldots, S_M$ respectively. Since no tax has been assumed, the stock prices just before going ex-dividend will be $S_1 + D_1, S_2 + D_2, \ldots, S_M + D_M$.

When dividends are expected, as the case of American call option, we can no
longer assert that the convertible option will not be voluntarily exercised prior to maturity. Sometimes, it will be optimal for the holders to voluntarily convert CCB just before an ex-dividend date. This is because the dividend payment causes the underlying stock price to jump down, making CCB less attractive abruptly. Obviously, at other times the holders should still follow the optimal conversion strategies in the section 3.3. Moreover, dividend payments have no effect on the optimal call policy in the section 3.4.

We start by considering the possibility of voluntary conversion just prior to the final ex-dividend date before the maturity of CCB. If the holders convert, they would receive \( \left( \frac{B_F}{P_1} \right) \left[ S_M + D_M \right] \). If they do not, the stock price would jump down and they would receive \( CCB \left( S_M, T - \tau^d_M; P_2 \right) \). Hence, if the former is greater than the latter, that is,

\[
\left( \frac{B_F}{P_1} \right) D_M > CCB \left( S_M, T - \tau^d_M; P_2 \right) - \left( \frac{B_F}{P_1} \right) S_M
\]

(25)

it must be optimal for the holders to voluntarily convert just prior to the final ex-dividend date. Similarly, for any \( i < M \), if

\[
\left( \frac{B_F}{P_1} \right) D_i > CCB \left( S_i, T - \tau^d_i; P_2 \right) - \left( \frac{B_F}{P_1} \right) S_i
\]

(26)

it must be optimal to voluntarily convert just prior to the ex-dividend date \( \tau^d_i \). The right side of these inequalities (25) and (26) is the difference immediately after going ex-dividend between the value of CCB and its conversion value. This difference is so-called time value of CCB and decreases with the underlying stock price increasing and the remaining time to maturity decreasing. Therefore, these inequalities will tend to be satisfied only when the underlying stock price is sufficient high and/or the
ex-dividend date is fairly close to the maturity of CCB.

7. Conclusion

This paper presents an equivalent decomposition method for valuing and analyzing the callable convertible bonds (CCB). Based on this method, the callable convertible discount bond can be completely replicated with its corresponding ordinary discount bond and three kinds of regular exotic options; the coupon-bearing callable convertible bond can be completely replicated with its corresponding ordinary bond and five kinds of exotic options. Obviously, these will be very helpful to understand the value components of callable convertible bonds and to replicate and hedge them.

Furthermore, the analytic formulae for CCB have been obtained and validated by comparing with Monte Carlo simulation. Obviously, these formulae can save huge computational resources that required by numerical procedures. Moreover, although these formulae seem complicated, the required assumptions about capital market and parameter estimations are the same as the Black-Scholes option pricing formulae. Therefore, widespread use of these formulae in practice would be expected. For example, it can be used for risk management, especially for running a large portfolio.

In addition, we analyze in detail the effects of coupon clauses, call clauses, soft call condition clauses and dividend payments on the value of CCB. These give a lot of new insights into the analysis of various callable convertible bonds. A useful direction for future research is to analyze the impacts of other clauses or factors, such as put
Appendix A:

Proof: Consider two investment portfolios: Portfolio I consists of one unit of CCB; Portfolio II consists of \((B_F/P_1)\) units of shares of the underlying stocks. Since no dividend has been assumed, Portfolio II always consists of \((B_F/P_1)\) units of shares of the underlying stock.

If no call were to be announced prior to maturity, from the inequality (3), prior to maturity Portfolio I would be worth at least as greater as Portfolio II even if there is no coupon. At maturity, in terms of the equality (7), the payoffs to these two portfolios are compared in Table 1. Table 1 shows clearly that Portfolio I is worth more than Portfolio II unless the holders voluntarily convert in which case they have the same value if there is no coupon.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Current value</th>
<th>stock price at maturity</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(CCB(S_0,T;C))</td>
<td>(B_F + Fv(T;C)) (\geq (B_F/P_1)S_T + Fv(T;C) - C_N)</td>
</tr>
<tr>
<td>II</td>
<td>((B_F/P_1)S_0)</td>
<td>((B_F/P_1)S_T)</td>
</tr>
</tbody>
</table>

Relationship between terminal values of Portfolio I and II

- \(V_I > V_{II}\)
- \(V_I \geq V_{II}\)

If a call were to be announced prior to maturity when the underlying stock price
is $S^*_r$, from the equality (6), Portfolio I would be worth $\max\left[(B_F/P_I)S^*_r, B_c\right]$. The payoffs to these two portfolios at the call announcement are compared in Table 2. Table 2 clearly shows that Portfolio I will never be worth less than Portfolio II and in some cases will be worth more, even if there is no coupon.

### Table 2. Demonstration that at the call announcement the payoff to Portfolio I will be at least as great as the payoff to Portfolio II.

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Current value</th>
<th>stock price at the call announcement</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$CCB(S_0,T;C)$</td>
<td>$(B_F/P_I)S^<em>_r \geq B_c$ \quad $(B_F/P_I)S^</em>_r &lt; B_c$</td>
</tr>
<tr>
<td>II</td>
<td>$(B_F/P_I)S_0$</td>
<td>$(B_F/P_I)S^*_r$ \quad $V_1 \geq V_2$</td>
</tr>
<tr>
<td>Relationship between the values of Portfolio I and II</td>
<td>$V_1 &gt; V_2$</td>
<td></td>
</tr>
</tbody>
</table>

Furthermore, prior to maturity or the call announcement, Portfolio I receive coupons but Portfolio II does not receive any cash. To sum up, both conditions for dominance defined by Merton (1973) exist. Hence, unless the current value of Portfolio I exceeds the current value of Portfolio II, i.e. $CCB(S_0,T;C) > (B_F/P_I)S_0$, the former will dominate the latter. Obviously, CCB should never be voluntarily converted except at maturity or the call announcement.

### Appendix B:

**Proof:** Suppose that this theorem is not the case.

From the inequality (5), both the decline of interest rates and the rise of the
underlying stock price can increase the lower limit of CCB. However, since the flat term structure has been assumed, only the latter is relevant here. From (5) again, it is very clear that the lower limit will approach the upper limit with the underlying stock price increasing. Therefore, the optimal call policy must yield a critical stock price \( S^*_\tau \) so that it is optimal for the issuers to announce a call as soon as the underlying stock price reaches \( S^*_\tau \). From the inequality (5), we can know \( S^*_\tau \leq (B_c / B_P) P_1 \).

Assume that it is optimal for the issuers to announce a call as soon as the underlying stock price reaches \( \bar{S}_\tau < (B_c / B_P) P_1 \). Let \( \bar{\tau} \) denote the time at which the underlying stock price reaches \( \bar{S}_\tau \) for the first time. According to this assumed optimal call policy, if \( \bar{\tau} < T \), the issuers will announce immediately a call at time \( \bar{\tau} \).

From the equality (6) together with \( \bar{S}_\tau < (B_c / B_P) P_1 \), the holders must choose to accept the call price in cash when the issuers announce a call.

\[
CCB(\bar{S}_\tau, T - \bar{\tau}; C) = \max \left[ \left( B_c / P_1 \right) \bar{S}_\tau, B_c \right] = B_c \quad \text{(B1)}
\]

On the other hand, assume that the issuers do not follow the assumed optimal call policy and will announce a call as soon as the underlying stock price reaches \( (B_c / B_P) P_1 \). Let \( \hat{\tau} \) denote the time at which the underlying stock price reaches \( (B_c / B_P) P_1 \) for the first time. Due to \( (B_c / B_P) P_1 > \bar{S}_\tau \), we must get \( \bar{\tau} < \hat{\tau} \). From the inequity (5), we can obtain at time \( \bar{\tau} \)

\[
\max \left[ B(T - \bar{\tau}; C), (B_c / P_1) \bar{S}_\tau \right] \leq CCB(\bar{S}_\tau, T - \bar{\tau}; C) \leq B_c \quad \text{(B2)}
\]

From Barone-Adesi, Bermudez and Hatgioannides (2003), the equality, \( (B_c / P_1) \bar{S}_\tau = CCB(\bar{S}_\tau, T - \bar{\tau}; C) = B_c \), is valid only when \( \bar{S}_\tau = (B_c / B_P) P_1 \). However, at fact \( (B_c / B_P) P_1 > \bar{S}_\tau \). Hence, if the issuers announce a call as soon as the
underlying stock price reaches $S^*_c = (B_c / B_r) P_1$, we can obtain

$$CCB\left(\bar{S}_c, T - \tau, C\right) < B_c$$  \hspace{1cm} (B3)$$

From (B1) and (B3), we can know that the assumed optimal call policy can not result in the minimum price for CCB, so it is not optimal. Hence, it must be optimal for the issuers to call CCB back as soon as the underlying stock price reaches $S^*_c = (B_c / B_r) P_1$.

**Appendix C:**

Let $U$ denote the set of the paths where the underlying stock price will reach the critical value $P_2$ from below prior to maturity. Let $V$ denote the set of the paths where the underlying stock price at maturity will exceed $\left(1 + \frac{C}{P_r}\right) P_1$. In this way, the set $U$ can be expressed as $U = \{ \tau^* \leq T \}$, the set $V$ as $V = \{ S_T > \left(1 + \frac{C}{P_r}\right) P_1 \}$, the intersection $\bar{UV}$ as $\bar{UV} = \{ \tau^* > T, S_T > \left(1 + \frac{C}{P_r}\right) P_1 \}$ and the intersection $\bar{\bar{UV}}$ as $\bar{\bar{UV}} = \{ \tau^* > T, S_T \leq \left(1 + \frac{C}{P_r}\right) P_1 \}$. Obviously, the first, second and third case of the ending of CCB corresponds to the set $U$, $\bar{UV}$ and $\bar{\bar{UV}}$ respectively. Let $1(A)$ denote the indicator function of the set $A$. Then, it’s easy to get

$$E\left[ 1(U) + 1(\bar{UV}) + 1(\bar{\bar{UV}}) \right] = 1$$ \hspace{1cm} (C1)$$

In the risk-neutral world, the payoffs to the corresponding ordinary bond and exotic options decomposed from CCB can be expressed respectively as follows.

$$Pv(T; C) = \sum_{i=1}^{N} C_i e^{-r_i t}$$ \hspace{1cm} (C2)$$

$$Fv(T; C) = \sum_{i=1}^{N} C_i e^{(T-t_i)} = e^T \sum_{i=1}^{N} C_i e^{(T-t_i)} = e^T Pv(T; C)$$ \hspace{1cm} (C3)$$
In the risk-neutral world, the payoffs to CCB are a lot more complex than these exotic options above. If the first case of its ending happens, its present value can be expressed as $e^{-r\tau} \left( (B_F / P_1) P_2 + Fv(\tau^*; C) \right)$. If the second case happens, its present value is $e^{-r\tau} \left( (B_F / P_1) S_T + Fv(T; C) - C_N \right)$. If the third case happens, its present value is $e^{-rT} \left( B_F + Fv(T; C) \right)$. Hence, the total payoffs to CCB can be expressed as follows.
Substituting the equations (C1) through (C12) into the equation (C13) yields

\[
CCB(S_0,T;C) = (B_f / P_i)P_2E^\mathbb{Q}\left[ e^{-r^*}1(U) \right] + E^\mathbb{Q}\left[ Fv(\tau^*;C)e^{-r^*}1(U) \right]
\]

\[
+ e^{-rT}E^\mathbb{Q}\left\{ \left( (B_f / P_i)S_T + Fv(T;C) - C_N \right)1(\bar{UV}) \right\}
\]

\[
+ e^{-rT}E^\mathbb{Q}\left[ (B_f + Fv(T;C))1(\bar{U}\bar{V}) \right]
\]

Therefore, Theorem 3 holds in the risk-neutral world. According to the risk-neutral valuation principal, Theorem 3 still holds even if the assumption of the risk-neutral world is relaxed.

Appendix D:

From Rubinstein and Reiner (1991b), we can get the expression of
Based on this expression, the analytic formula for
\( A_{BC}^t\left(S_0,T;F_v(\tau^*;C),P_2\right) \) can be derived below.

\[
A_{BC}^t\left(S_0,T;F_v(\tau^*;C),P_2\right) = E^Q\left[e^{-\tau^*} \left( \sum_{k=1}^{\infty} C_k e^{(\tau^*-\tau_k)} \right) 1(\tau^* \leq T) \right] \quad \tau_k \leq \tau^* < \tau_{k+1}
\]

\[
= C_1 e^{-\tau_1} E^Q\left[1(\tau_1 \leq \tau^* < \tau_2) + \left( C_1 e^{-\tau_1} + C_2 e^{-\tau_2} \right) E^Q\left[1(\tau_2 \leq \tau^* < \tau_3) \right] \right] + \cdots
\]

\[
+ \left( \sum_{i=1}^{N-1} C_i e^{-\tau_i} \right) E^Q\left[1(\tau_{N-1} \leq \tau^* < \tau_N) \right] + \left( \sum_{i=1}^{N} C_i e^{-\tau_i} \right) E^Q\left[1(\tau^* = T) \right]
\]

\[
= \sum_{i=1}^{N-1} \left( C_i e^{-\tau_i} \right) E^Q\left[1(\tau_i \leq \tau^* < \tau_N) \right]
\]

\[
= \sum_{i=1}^{N-1} \left( C_i e^{-\tau_i} \right) \left( \Pr(0 < \tau^* < \tau_N) - \Pr(0 < \tau^* < \tau_i) \right)
\]

\[
= \sum_{i=1}^{N-1} \left( B_i R_i e^{-\tau_i} \right) \left[ \left( \frac{P_2}{S_0} \right)^{2\pi/\sigma^2} N(-a_3) + N(-a_4) \right.
\]

\[
- \left[ \left( \frac{P_2}{S_0} \right)^{2\pi/\sigma^2} N(-a_5) + N(-a_6) \right] \right]
\]

(D1)

References


