A Closer Look at Stable Value Funds Performance

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INTRODUCTION

There exists a paucity of academic literature on stable value (SV) funds, although a growing volume of industry and practitioner literature has provided an in-depth look at how the funds are managed and the guarantees secured.[1] To date, no rigorous analysis has been published on the performance of stable value funds from the investor's point of view. This is rather surprising, because the funds occupy a prominent place among retirement investment vehicles, approaching half a trillion dollars worth of stable value assets under management. They are available to participants in defined contribution (DC) savings and profit sharing plans, and Section 529 Tuition Assistance Plans. Stable value funds have grown in popularity and today are offered as an investment option in almost two-thirds of all DC plans, and constitute roughly one-third of the assets in these plans.[2]

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[1] See, for example, *Stable Times,* a quarterly publication of the Stable Value Investment Association, and *The Handbook of Stable Value Investments,* Frank J. Fabozzi, editor.

[2] Current statistics are available at www.stablevalue.org

In this study, we provide what we understand to be the first published analysis of the performance of stable value funds. Their performance is compared to basic investment alternatives such as U.S. large and small stocks, long-term government and corporate bonds, intermediate government bonds, and money market funds using three methods: mean-variance analysis, stochastic dominance analysis, and an enhanced multiperiod utility analysis. We show that since the inception of stable funds in late 1988, none of these other asset classes has dominated them; on the contrary, we show that stable value funds have dominated money market and intermediate government bond funds over a wide range of risk aversion levels, and when combined with stocks and long-term government bonds, occupy a prominent and often dominant part in optimal portfolios.

BACKGROUND

Stable value funds are labeled in various ways, including Capital Accumulation, Principal Protection, Guaranteed, Preservation, Income funds, as well as Guaranteed Investment Contracts (GICs), among others. The precursors to stable value funds have been around since the 1970s, coinciding with the development of U.S. defined contribution (DC) plans. From their outset, these funds were comprised largely of laddered maturity GICs issued by insurance companies. The guaranteed returns were backed only by the GIC issuer's claims-paying ability, which concerned some plan sponsors. The GICs were unattractive to others due to their lack of flexibility and control, as the assets backing the GIC contracts were owned and managed by the insurer. Such concerns were alleviated, in part, by the creation of separate account GICs, where the assets underlying the contracts were held in a separate account and could not be used to discharge claims against the general account of the insurer. Then, in mid-1988, when the broader stable value funds market sprung up, synthetic GICs were created in an effort to allow plans to retain legal title to plan assets, and to provide additional flexibility in terms of investment strategy and asset selection.[3]

[3] Traditional GICs, are issued by an insurance company that guarantees the principal invested and pays a periodically-reset interest rate for a certain period of time. In contrast with a traditional GIC, whose sole guarantor is the issuing insurance company, a synthetic GIC contract's guarantees are "based on a separately managed underlying portfolio of fixed securities owned by the plan. Synthetic GICs are

A stable value fund offers principal protection and liquidity to individual investors, and a quarterly guaranteed rate of return commensurate with yield levels on intermediate-term bonds. However, over ensuing three-month intervals, the guaranteed rate of return moves more slowly than intermediate yields. As a consequence of these features and the underlying intermediate-term bond investments, stable value funds provide to investors returns with very small volatility. This combination of bond-yield-like returns and low volatility elicits contract or book value accounting of the investment, and is achieved by means of a crediting rate (see below) that allows the fund to smooth returns over time, in spite of fluctuations in value of the underlying bond portfolio.[4]

Today, traditional GICs constitute 15%-20% of the overall stable value fund assets in the aggregate, while the remainder is comprised of synthetics, which themselves are comprised of high quality corporate and government intermediate bonds and mortgage-backed securities, and whose portfolio is protected against interest rate risk through a "wrap" obtained through a high quality financial institution. Traditional GICs, however, continue to represent an important segment of the entire stable value marketplace.

Stable value funds do not require a set holding period and generally provide full access to the participant's principal and accumulated interest without a penalty. However, they are subject to the general restrictions within the overall plan. For example, some plans restrict participants against the direct transfer to a competing fixed income or money market fund by requiring that money transferred out of stable value be first invested in a stock fund for a short period such as a month. Together with the fact that plan participants make individual decisions regarding the allocation of their funds among various alternatives and do not act in concert, this allows the wrapper guarantees to be purchased for a fraction of what it would cost if interest arbitrageurs dominated the pool and were revising their allocations aggressively.

sometimes called 'wrappers' or 'wrap contracts' because they 'wrap' a specific portfolio with contractual benefits" (Tobe, 2004, p. 82).

[4] See, SVIA (2005); Tobe (2004).

CREDITING RATE FORMULA

From an investor's viewpoint, stable value funds operate similarly to a passbook savings account. They accrue interest at a prespecified crediting rate that is generally updated every three months to reflect changing market conditions. Their principal is secure and grows over time by the amounts of interest credited to their account. Crediting rates on stable value funds change more slowly than bond yields and are computed according to a formula which basically produces an internal rate of return for the investment by requiring that the contract (or book) value of the portfolio converge to its market value by the end of the assumed duration . We define the variables in the formula as follows:

CR: crediting rate applied to the accounts of investors in the stable value funds

MV: market value of the underlying portfolio

CV: Contract value of the underlying portfolio

D: duration of the underlying portfolio or duration of a benchmark portfolio

Y: yield of the underlying portfolio, as described further below.

Given these variables, the future market value (*FMV*) of the portfolio is given by

$$FMV = MV(1+Y)^{D}, \tag{1}$$

and the crediting rate that guarantees this value, given the current contract value, is the solution to

$$FMV = CV(1+CR)^{D}. \tag{2}$$

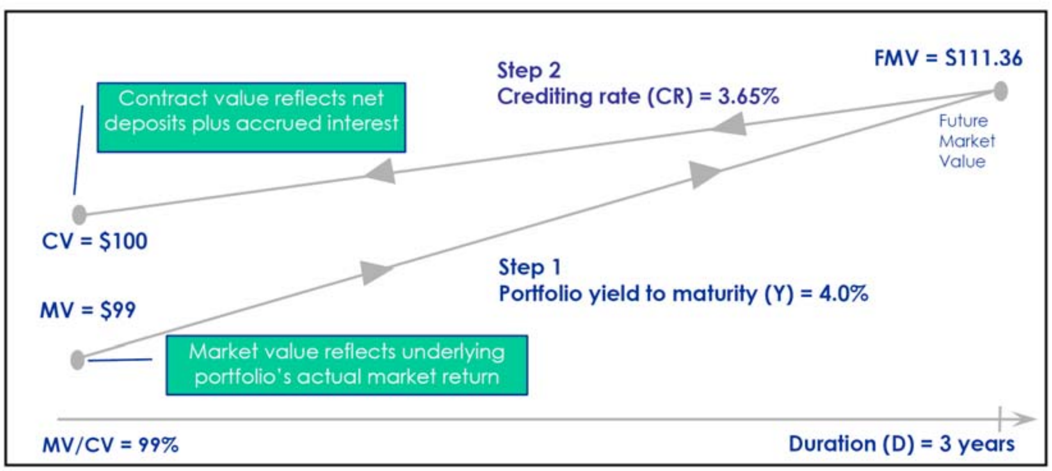
Therefore, the crediting rate formula is given by equating the right-hand-sides of expressions (1) and (2), and solving for CR,

$$CR = (1+Y)\left(\frac{MV}{CV}\right)^{1/D} - 1. \qquad (3)$$

Although other variations of the crediting rate formula are also used, expression (3) is the one most generally used.[5] Apart from small differences in the crediting rate formulae, there are differences in the way different managers calculate the inputs to the formulae. For example, with respect to the measure of duration *D*, some fund managers use the duration of a benchmark portfolio, while others use the duration of the underlying securities. The yield measure *Y* is most often a duration market-weighted bond equivalent yield, although some variations have occurred.

Illustration 1 below details how the crediting rate is designed to make the contract value of the portfolio converge to its market value over the duration of the portfolio, assuming market conditions do not change in the interim.

Illustration 1: Setting Synthetic GIC Crediting Rates



Step 1
Calculate estimated future market value by compounding the current market value at the portfolio's current yield to maturity for a period equal to the portfolio's duration.

Step 2
Determine new crediting rate as the discount rate that equates the estimated future market value with the current contract value.

Source: SVIA

[5] See, SVIA (2005, p. 4); Tobe (2004, p. 9).

As indicated above, the crediting rate effectively smoothes returns by distributing gains and losses over a period of time related to the duration of the portfolio. The crediting rate formula above implies that the crediting rate is between the portfolio's return and its yield, *Y*, and closer to the portfolio's yield the longer the duration, *D*, is. The important thing to remember is that individual investors receive the same rate of return as the stated crediting rate, since principal is guaranteed.

PERFORMANCE MEASUREMENT

In this study we will measure the performance of stable value funds vis-à-vis money market, intermediate government bond, long-term government and corporate bond, small stocks, and large stocks investments using three methods of analysis: mean-variance analysis, stochastic dominance analysis, and enhanced multiperiod utility analysis. Each method has its advantages and drawbacks, but together we get a fairly clear picture of how well stable value funds have performed.

We begin with a mean-variance analysis, owing to its simplicity and ubiquitous use in practice more than its theoretical properties.[6] Strictly speaking, the validity of this approach depends entirely upon whether investors can and do consider variance to be an adequate measure of investment risk – in other words, investor preferences are satisfactorily modeled using quadratic utility.

Beginning as early as 1967, Arditti determined that investors considered measures of downside risk beyond variance, and countless additional studies along similar lines have continued until now to demonstrate that variance is an inadequate measure of either security or portfolio risk.[7] However, if market returns can be fully described by the first two moments of their distribution, then restricting one's performance analysis to a mean-

[6] An excellent treatise on this approach is provided by Markowitz (1987).

[7] Indeed, as reported by Douglas (1969), John Lintner's initial cross sectional tests conducted in 1965 found that residual risk, which according to the Capital Asset Pricing Theory's version of mean-variance analysis is not supposed to be priced by the marketplace, was indeed important to investors. More rigorous studies since then have reconfirmed these early findings. Most recently, Cvitanic, Polimenis, and Zapatero (2007) have found that ignoring higher moments can lead to significant overinvestment in risky securities, especially when volatility is high.

variance analysis can be justified, even if investors would otherwise be concerned about higher (and non-existent) moments of the return distribution. But all tests with which we are familiar demonstrate that return distributions for stocks, bonds, and money market instruments cannot adequately be characterized by their means and variances, nor does modified Brownian motion fully capture the movement in these asset returns.

Accordingly, we next measure investment performance using stochastic dominance analysis. Introduced in 1969 by Hanoch and Levy and by Hadar and Russell to remedy the shortcomings of mean-variance analysis, stochastic dominance approaches have the clear advantage of taking into account all moments of the return distributions, and providing investment dominance analyses that do not depend upon knowing the exact shapes of investor preference functions. This has another distinct advantage over the mean-variance approach, which cannot be valid for various horizons simultaneously because it relies on normally distributed returns, which if valid (under certain conditions) for single-period returns is not valid for multiperiod returns. By contrast, the stochastic dominance approach remains valid because it is distribution-free. The limitations and additional virtues of this approach are discussed at length in the authoritative treatise by Levy (2006). While some of the limitations have been overcome by a plethora of research, beginning in the 1970s and continuing up until today, there remain two:

1) Stochastic dominance methods do not provide guidance into the construction of a portfolio from various individual securities.
2) Stochastic dominance methods do not provide an equilibrium price for securities.

Our third approach, the discrete-time multiperiod investment theory of Mossin (1968), Hakansson (1971, 1974), Leland (1972), Ross (1974), and Huberman and Ross (1983), remedies the failings of mean-variance analysis as well as the limitations of stochastic dominance analysis at the high cost of specifying the form of the intertemporal preference function. This theory has been applied to the asset allocation problem with some success, by Grauer and Hakansson (1982, 1985, 1986, 1987, 1993, and 1995) and others, where an empirical probability assessment approach was used to implement a set of investment strategies. We will not rehearse the details of the methodology here, as they are well documented in Grauer and Hakansson (1986).

Our calibration periods were comprised of 40 consecutive quarters, rolled forward quarter by quarter over time, similar to what Grauer and Hakansson used in most of their studies. However, we did undertake an enhancement to their approach, by inserting for each quarter the then current expected returns for each asset class rather than using their historical mean returns. This is especially important for interest-bearing securities, because the lagging 40-quarter average returns may not reflect yield conditions and expected returns for the current quarters. Also, for stocks, rather than use historical average excess returns, we instead conducted a sensitivity analysis using various spreads above Treasury bills to test for the effect of different expected returns.[8] Then we adjusted the rolling time series of lagging quarters by using upper moment and co-moment preserving spreads, while substituting expected returns for only the first moment. Clearly this is more realistic and takes into account the available information on expected returns for each quarter across all asset classes, while taking full advantage of the relative stability of upper moments and co-moments of their distributions. We then derived optimal asset allocations for a wide range of risk aversion levels.

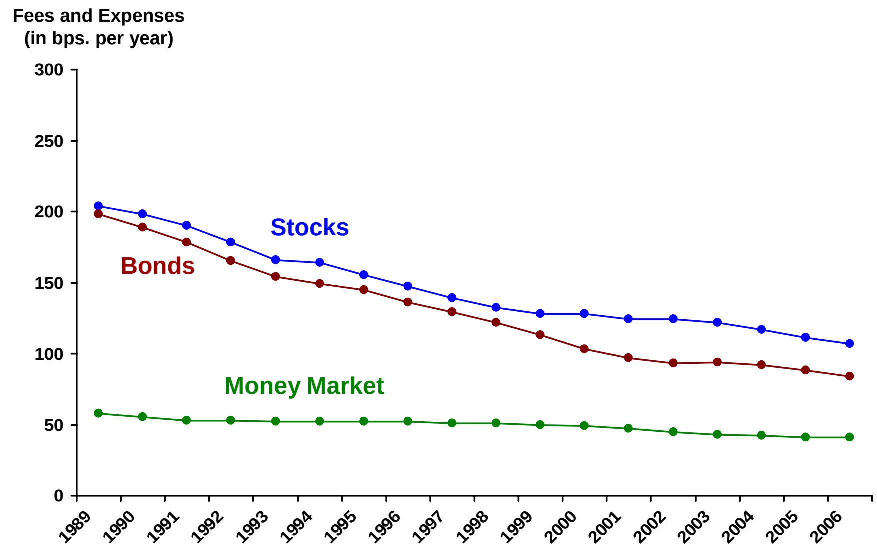
DATA

We begin our analysis in January 1989, just a few months beyond the inception of SV funds, and continue through December 2006. For non-SV investments, we use total monthly returns on the S&P 500 index, Ibbotson's small stocks index, the Lehman Intermediate Government/Credit index, and the Merrill Lynch 3-month Treasury Bill index (supplemented for the year 2006 with monthly total returns on the Lehman Bellwether 3-month Treasury index). For SV funds, we use total net monthly returns and asset values (quarterly where monthly data are not available) on various SV fund families sponsored by members of the Stable Value Investment Association (SVIA.)

[8] See Merton (1980) for a discussion of expected market returns.

We note that, except for small stocks and SV funds, returns are gross returns and need to be adjusted for management fees and transaction costs to be comparable to returns on small stocks and SV investments. We do this by subtracting average fees and expenses reported by the Investment Company Institute (ICI) for stock, bond and money market funds over the period of our analysis from the corresponding large stocks, bond or money market returns.[9] Figure 1 shows the evolution of mutual fund fees and expenses for the period of our analysis.

Figure 1. Annual Fees and Expenses for Stock, Bond and Money Market Funds



Source: ICI.

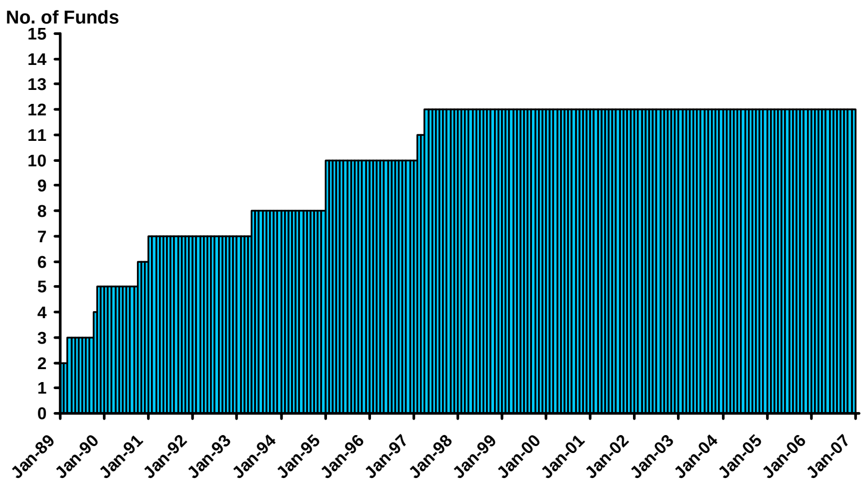
We have net return data on a subset of twelve major SV fund complexes, although some of the funds began operations after January of 1989.[10] Figure 2 shows the number of

[9] For annual fund fees and expenses from 1980 through 1999, see *Fundamentals, Investment Company Institute Research in Brief*, Vol. 14, No. 6, October 2005, pp. 3, 6, 7; for 2000-2004 fees and expenses, and for 2005 money market fees, see *ICI Research Fundamentals*, Vol. 15, No. 4, June 2006, p. 3; for 2005-2006 fees and expenses for stock and bond funds, and 2006 fees for money market funds, see *ICI 2007 Factbook*, Figures 5.1 and 5.4, pp. 48, 51. For a given year, we subtract one twelfth of the stock, bond or money market fees and expenses from the corresponding stock, bond or money market monthly returns in that year to calculate net monthly returns.

[10] Two of the twelve return series are quarterly net returns. Since the return on an SV fund is a quarterly crediting rate, set prior to the quarter to which it applies, for those two return series we use this quarterly rate to estimate a monthly return. For each month in a given quarter, we calculate the monthly return as

major SV fund complexes (among the 12 we obtained) for which return data exist on each month in the period of interest. The number of fund complexes reporting their return data over time mirrors the growth in the number of fund complexes in the entire market and is considered representative of the overall population of stable value funds. We observe that, as the industry expands, more and more plan sponsors offer SV investment options.

Figure 2. Number of Funds by Month in the SV Average Return Indices



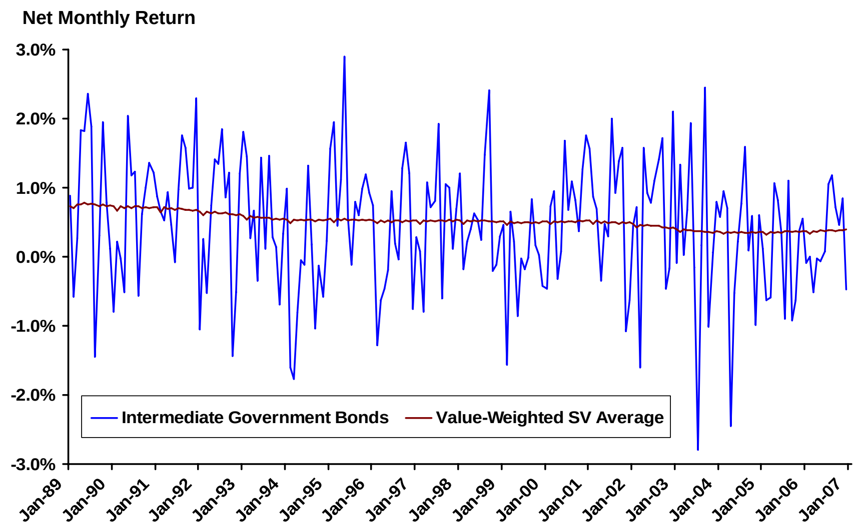
Source: SVIA.

Using these data as well as asset values for each fund complex, we construct a value-weighted average return series.[11] Figure 3 plots this series together with the intermediate government bond net monthly return series for the period of our analysis.

$R_{Monthly} = (1 + R_{Quarterly})^{(\text{Days in Month} \div \text{Days in Quarter})} - 1$. This method results in a days-in-month pattern that very closely follows the pattern observed in the other SV monthly return series we obtained.

[11] We also used an equal-weighted average and obtained essentially the same overall results.

Figure 3. Intermediate Government Bond and Stable Value Return Indices



Source: Morningstar (Ibbotson) and authors' calculation based on data provided by SVIA.

The individual funds return series in the SV average are highly correlated among themselves. Both the average of pairwise correlation coefficients and an efficient measure of multiple correlation, the multirelation coefficient, indicate that the SV value-weighted average return series is highly representative of the individual fund returns.[12] Table 1 reports these coefficients for a set of all seven SV fund complexes with data from January of 1991, and for the full set of twelve fund complexes with data from April 1997.

Table 1. Correlation Among the Return Series in the SV Industry Averages

	7 Funds Jan-91 — Dec-06		12 Funds Apr-97 — Dec-06	
	Levels	Differences	Levels	Differences
Multirelation Coefficient:	**99.5%**	**97.7%**	**99.7%**	**98.3%**
Pairwise Correl. Coeff. (average):	**96.8%**	**81.9%**	**93.4%**	**78.0%**

Note : The multirelation coefficient is one minus the smallest eigenvalue of the correlation matrix of the data.

[12] The multirelation coefficient and its calculation are introduced and described in Zvi Drezner (1995). A significance level test of the coefficient is provided by Dear and Drezner (1997).

We observe that the return series for the fund complexes comprising the index are highly correlated, even when first differences are used to eliminate the downward trend in the data.

Table 2 presents summary statistics for the seven asset classes we study.

Table 2. Summary Statistics - Monthly Net Returns, Jan-89 through Dec-06

	Large Stocks	Small Stocks	Long-Term Gov't Bonds	Long-Term Corp. Bonds	Interm. Gov't Bonds	Money Market	Stable Value
N	216	216	216	216	216	216	216
Mean	0.90%	1.33%	0.66%	0.63%	0.46%	0.34%	0.52%
STDEV	4.01%	5.53%	2.53%	2.06%	0.94%	0.17%	0.12%
Min	-14.57%	-20.10%	-9.90%	-8.89%	-2.80%	0.03%	0.32%
Max	11.28%	23.58%	7.78%	6.19%	2.90%	0.76%	0.78%
Jarque-Bera	47.14	16.55	44.72	30.50	66.16	106.03	116.22
JB p-value	0.0000	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000

Note: The Jarque-Bera statistic tests for the null hypothesis of normality, under which it has a Chi-square distribution with two degrees of freedom. 95%, 99% and 99.9% critical values are, respectively, 5.99, 9.21 and 13.82.

Table 2 shows that, over the period of January 1989 through December of 2006, SV investments have had, on average, a higher net monthly return and a lower return volatility than either money market or intermediate government funds. As expected, when compared to stocks or long-term bonds, SV funds have exhibited both lower average returns and volatility. These facts lie behind the results that we present in the next section.

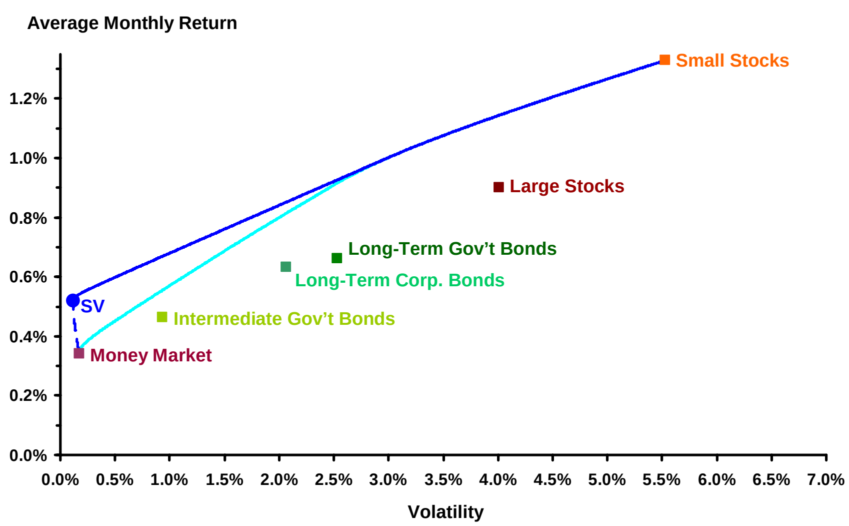
RESULTS

Mean-variance analysis. As indicated earlier, mean-variance analysis provides a characterization of the trade-off between risk and return that is neither supported by the statistical properties of the return data, nor by the theoretical logic of risk-aversion. Despite these shortcomings, the mean-variance approach provides useful insights into the ability of SV investments to dominate other asset classes.

In this section we present evidence supporting the conclusion that, even as stand-alone investments, SV funds are superior in the mean-variance sense to money market and intermediate government bond funds. We also show based solely on historical returns that, when included in optimal mean-variance portfolios, SV funds contribute significantly to the portfolio, to the exclusion of money market, intermediate government bonds, long-term corporate bonds and even large stocks. In other words, optimal mean-variance portfolios contain only SV funds, long-term government bonds and small stocks in proportions that naturally vary with the expected return (or, alternatively, the expected volatility) of the optimal portfolio.

When discussing summary statistics for our net monthly return data in Table 2 above, we observed that, over the period of our study from January 1989 through December 2006, SV returns exhibited both a higher mean and lower volatility that either money market or intermediate government bond returns. This feature can be seen in Figure 4, where we plot two efficient frontiers, one including all seven asset classes in our study, and one that excludes SV funds.

Figure 4. Efficient Frontiers for Alternative Asset Classes



It is interesting to note the large scope for improvement that inclusion of SV investments brings to an optimal mean-variance portfolio for approximately the lower half of the expected return range. As revealing as Figure 4 is, it does not show the full extent to which SV investments contribute to an optimal portfolio since it says nothing about the relative allocations of wealth among SV funds and other investments at different points along the efficient frontier. Table 3 reports these optimal weights for selected expected returns ranging from 0.52%, the historical average net monthly return for SV funds, to 1.33%, the historical small stocks net monthly return.

Table 3. Optimal Weights across Alternative Asset Classes for Mean-Variance Efficient Portfolios

	Asset Class						
Expected Return	Large Stocks	Small Stocks	LT Gov't Bonds	LT Corp. Bonds	Interm. Gov't Bonds	Money Market	SV
0.52%	0	0.2%	0	0	0	0.7%	99.1%
0.56%	0	4.3%	3.9%	0	0	0	91.8%
0.60%	0	8.6%	8.0%	0	0	0	83.4%
0.64%	0	12.9%	12.1%	0	0	0	75.0%
0.68%	0	17.1%	16.3%	0	0	0	66.6%
0.72%	0	21.4%	20.4%	0	0	0	58.1%
0.76%	0	25.7%	24.6%	0	0	0	49.7%
0.80%	0	30.0%	28.7%	0	0	0	41.3%
0.84%	0	34.3%	32.9%	0	0	0	32.9%
0.88%	0	38.5%	37.0%	0	0	0	24.4%
0.92%	0	42.8%	41.2%	0	0	0	16.0%
0.96%	0	47.1%	45.3%	0	0	0	7.6%
1.01%	0	51.5%	48.5%	0	0	0	0
1.05%	0	57.6%	42.4%	0	0	0	0
1.09%	0	63.7%	36.3%	0	0	0	0
1.13%	0	69.7%	30.3%	0	0	0	0
1.17%	0	75.8%	24.2%	0	0	0	0
1.21%	0	81.9%	18.1%	0	0	0	0
1.25%	0	87.9%	12.1%	0	0	0	0
1.29%	0	94.0%	6.0%	0	0	0	0
1.33%	0	100.0%	0	0	0	0	0

Note: Weights may not add up to 100% across a given row due to rounding.

We observe that no optimal mean-variance portfolio along the efficient frontier includes money market instruments (except less than 1% weight at expected return 0.52%), intermediate government bonds, long-term corporate bonds or even large stocks. We also observe that SV funds are predominant, in the sense of having the largest weight for expected returns, in the lower half of the expected return range, where one would conventionally anticipate seeing money market and intermediate government bond investments.[13] We do not want to ascribe too much importance to these findings, as they are derived in the context of return distributions that are decidedly not normal, as evidenced by the bottom row of Table 2. Accordingly, the theoretical support for the conclusiveness of our mean-variance analysis is compromised.

Stochastic dominance analysis. We next discuss the ability of SV funds to dominate alternative asset classes in the sense of stochastic dominance (SD) which, as we indicated above, provides dominance criteria under very general conditions with respect to an investor's attitudes toward risk.

First-degree stochastic dominance (FSD) imposes only one preference restriction – that investors prefer more wealth to less wealth. In addition to this requirement, second-degree dominance (SSD) requires investors to be risk averse that is, to dislike a drop in wealth more than they like a wealth increase of the same magnitude. The development of third-degree stochastic dominance (TSD) was motivated by a long observed preference among investors for positively skewed returns. A subset of the class of investors that prefer returns exhibiting third-degree stochastic dominance is the important group whose preferences are characterized by decreasing absolute risk aversion (DARA). Such investors are willing to pay less for insuring against a given sized risk, on average, as they accumulate greater wealth, which seems to accord with observed behavior toward risk. Fourth-degree stochastic dominance (4SD) was developed to capture investors' aversion toward kurtosis, where returns are characterized by peaked distributions and fat tails, such that losses can be extreme. Of course kurtosis can favor investors who have asymmetric claims toward returns, such as investors in call options, but for investors who

[13] We have also constructed efficient frontiers excluding long-term bonds and small stocks. In this case the results are even more in favor of SV investments: only SV and large stock funds are relevant in an optimal mean-variance portfolio, with the weights declining close to linearly from 100% at the SV expected return of 0.52% to 0% at the large stock expected return of 0.90%.

have equal claims to both tails of a distribution, the fatter tails cause a disproportionate loss in utility.[14]

Table 4 presents the SD results among the seven asset classes in our study. Only the SV fund historical returns distribution dominates any other in the stochastic dominance sense up to the fourth degree; none of the other asset classes dominate SV funds.

Table 4. Stochastic Dominance Among Alternative Asset Classes

Does the Column Investment Stochastically Dominate the Row Investments?

	SV	Money Market	Interm. Gov't Bonds	LT Corp. Bonds	LT Gov't Bonds	Large Stocks	Small Stocks
SV		NO	NO	NO	NO	NO	NO
Money Market	YES (4SD, TSD, SSD, FSD)		NO	NO	NO	NO	NO
Interm. Gov't Bonds	YES (4SD, TSD, SSD) NO (FSD)	NO		NO	NO	NO	NO
LT Corp. Bonds	NO	NO	NO		NO	NO	NO
LT Gov't Bonds	NO	NO	NO	NO		NO	NO
Large Stocks	NO	NO	NO	NO	NO		NO
Small Stocks	NO	NO	NO	NO	NO	NO	

Note: A cell with a single **NO** result indicates that the column investment does not SD the row investment for any of the first four degrees.

Turning to SV funds, they stochastically dominated money market investments by the first-degree and, as a corollary, by any other degree as well. This is a direct consequence

[14] See the detailed exposition in Levy (2006) for a complete characterization of the necessary and sufficient conditions for SD.

of the fact that, when sorted returns for SV and money market funds are compared in a pairwise fashion, the SV return was always greater than the corresponding money market return. In other words, the empirical cumulative distribution function (CDF) of the money market returns was strictly above and to the left of the empirical CDF of the SV returns, meaning that for any given return, the probability of obtaining a lower return with a money market fund is greater than with an SV fund and any investor who preferred more wealth to less wealth should have avoided investing in money market funds when SV funds were available, irrespective of risk preferences.

Although SV funds failed to stochastically dominate intermediate government bonds by first degree, they dominated by second and higher degrees. This result is, in turn, a direct consequence of the fact that while the empirical CDFs of these two asset classes cross (thus preventing first-degree stochastic dominance), positive intermediate bond returns during the period of our study were never large enough, relative to corresponding SV returns, to make at least some risk-averse investors prefer the riskier intermediate bond investment. Technically, the integral of the difference between the intermediate bond return distribution and the SV return distribution is positive for any return.

The results in this Sub-section are remarkable. Not surprisingly, there is no stochastic dominance of any one traditional class over another; indeed dominance is rarely encountered. So, it was surprising to find that stable value investments dominated two of the major traditional investment classes.

Intertemporal optimization analysis. The intertemporal investment model of Grauer and Hakansson (1982, 1985, 1986, 1987, 1993, and 1995) considers an investor who, at the beginning of each period, allocates wealth among various investment alternatives so as to maximize expected utility of wealth. The investment alternatives we consider are the same as in our previous analysis, but the investment horizon is assumed to be a quarter. Therefore, the return data we use are net quarterly returns for the period Q1-1989 through Q4-2006.

At the beginning of each quarter t, the investor chooses a portfolio that maximizes expected utility of wealth,

$$E[U(1+r_t)] = E\left[\frac{1}{a}(1+\sum_i w_{it}r_{it})^a\right] \tag{4}$$

Subject to $w_{it} \geq 0,$ for all i,t and $\sum_i w_{it} = 1,$ for all $t,$ where E is the expectations operator, w_{it} is the fraction of wealth allocated to investment i in period t, r_{it} is the investment i return that will obtain at the end of quarter t, and $a \leq 1$ is a risk parameter. The function $U(1+r_t) = \frac{1}{a}(1+r_t)^a$ is the familiar constant relative risk aversion (CRRA) utility function.

In order to evaluate the expectation in (4) we need the theoretical distribution of quarterly returns which is not known. We therefore approximate this expectation using observed return data for the 40 quarters prior to a decision quarter t. We also use yield data on money market and bond funds known at the beginning of quarter t, as explained below.

Note that SV funds guarantee a crediting rate that is generally reset at the beginning of each quarter. This means that for each decision period, the SV return is known at the time the investor solves the optimization problem. Therefore, the SV return is not random. [15] However, the remaining investments, large and small stock funds, long-term government and corporate bond funds, intermediate government bond funds and money market funds have random decision period returns at the beginning of the period and thus the expected utility of wealth needs to be estimated. Grauer and Hakansson (1982, 1985, 1986, 1987, 1993, and 1995) use the realized returns over the 32 or 40 quarters prior to each decision period in order to estimate the expected utility of wealth and solve the investor's optimization problem. We follow a slightly different approach, in order to avoid the

[15] We have ignored in our modeling here that a tiny part of the monthly return reflects return on cash and is therefore not entirely known at the beginning of the quarter. Moreover, large funds may have several overlapping cohort segments that constitute a given quarterly segment, and the cohorts may mature at different times during the quarter. As they roll off, they are substituted by a new cohort segment. In the past, this substitution has given rise to as much as an 8 b.p. change in the overall returns of a given quarter. These changes are too small to affect the results of this section, so they are not considered here.

difficulties involved in estimating expected returns using historical data (Merton, 1980).[16]

In the case of bond and money market funds, we use available information on long-term government bond yields, intermediate-term government bond yields, and money market yields at the beginning of each decision period, converted to net quarterly yields, as estimates of the corresponding expected bond and money market returns. And since calculating the expected utility in expression (4) above involves the full joint distribution of the various investments, we add this estimate of a given expected return to the difference between the actual returns and their sample mean over the 40 quarters prior to the decision period. For each bond or money market fund, this procedure gives us a 40-quarter series that preserves the same second and higher moments and co-moments as the original quarterly return data, and has a sample mean that is the corresponding net quarterly yield.

For large and small stocks we use a similar approach concerning the second and higher moments, but the expected return over the decision quarter is the sum of the concurrent money market yield, expressed as a net quarterly yield, plus a large or small equity premium. We consider two alternative values for the large stock equity premium, 2.5% and 3.5% per year over the money market expected return calculated as described above. These values reflect the current discussion on the equity premium that is likely to obtain over the foreseeable future (Siegel, 2005). Concerning the small stock premium, we assume that it will be in the same proportion to the assumed large stock equity premium (either 2.5% or 3.5%) as the historical small stock premium has been to the historical large stock premium over the period 1989 through 2006. This approach results in a small stock premium of 4.59% corresponding to a large stock premium of 2.5%, and a small stock premium of 6.44% corresponding to a large stock premium of 3.5%.

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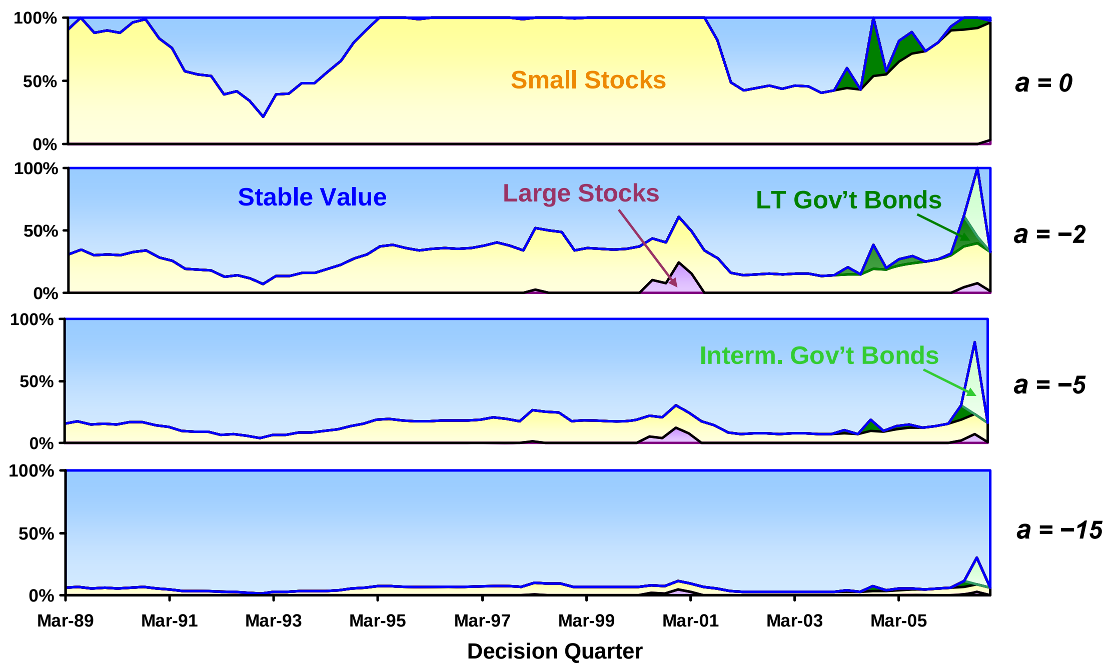
[16] Properly speaking, solving the investor's maximization problem does not require direct estimation of expected returns. However, the sample mean of expected utility in expression (4), based on data observed over the 40 quarters prior to the decision period, will indirectly inherit the problems associated with estimation of expected returns, especially during a period where expected asset returns may have been time-varying.

Optimal Asset Allocation Over Time. We first find optimal weights for all seven investment alternatives, by solving the optimization problem in expression (4) for each of the 72 quarters in the period 1989 through 2006, using empirical distribution of returns as described above in order to calculate expected utility of wealth. Figure 5 shows the optimal portfolio weights for each of the 72 quarters and for selected values of the risk parameter *a* and large stock equity premium of 3.5% per year.

Figure 5. Asset Allocation over Time across Four Levels of Risk Aversion (Large Stock Premium = 3.5%)

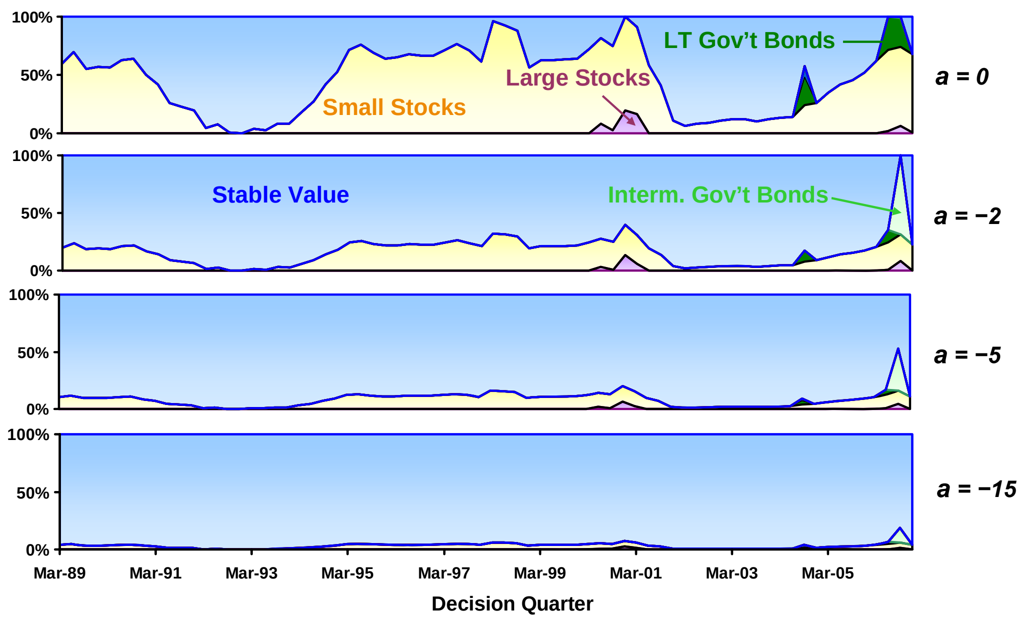


We observe that, in general, small stocks and stable value investments predominate throughout the period, and that the more risk-averse investors are the larger is the proportion of stable value in their portfolios. Exceptions occurred briefly during the late nineties when large stocks reached up to 24% of the portfolio in Q4-2000, and during the last few years when long-term and intermediate government bonds entered into the optimal asset allocation.

Figure 6 shows portfolio weights for a large stock premium of 2.5%, with patterns very similar to those in Figure 5, but stocks are weighted less in favor of SV across all levels of risk aversion, as would be expected for smaller equity risk premia.

It is worth noting that for moderate values of the risk parameter, between −2 and −5, SV investments generally have the largest weight in an optimal portfolio for all quarters in our analysis except for Q3-2006.

Figure 6. Asset Allocation over Time across Four Levels of Risk Aversion (Large Stock Premium = 2.5%)



Optimal Asset Allocation for Q4-2006. We next look at the adequacy of using 40 quarters of data in the calculation of expected utility by solving for the investor's optimal allocation at the beginning of the fourth quarter of 2006 using successively longer calibration windows, beginning with 40 quarters prior to Q4-2006 and going all the way to 111 quarters, from Q1-1979 through Q3-2006. Figures 7 and 8 show the optimal allocations for Q4-2006 corresponding to successively longer calibration windows for large stocks equity premium of 3.5% and 2.5% per year, respectively.

Figure 7. Asset Allocation for Q4-2006 across Four Levels of Risk Aversion (Large Stock Premium = 3.5%)

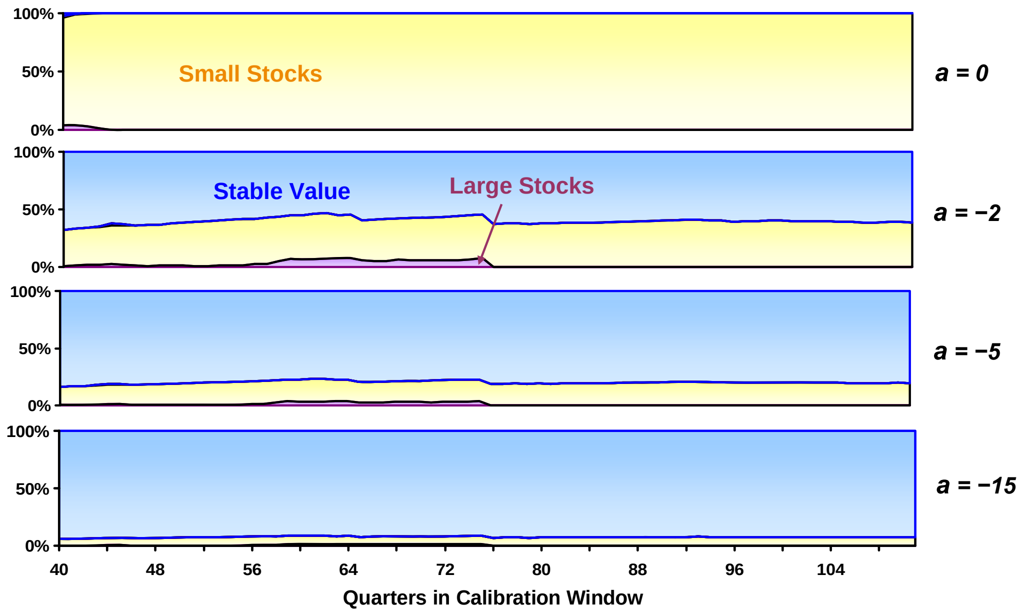
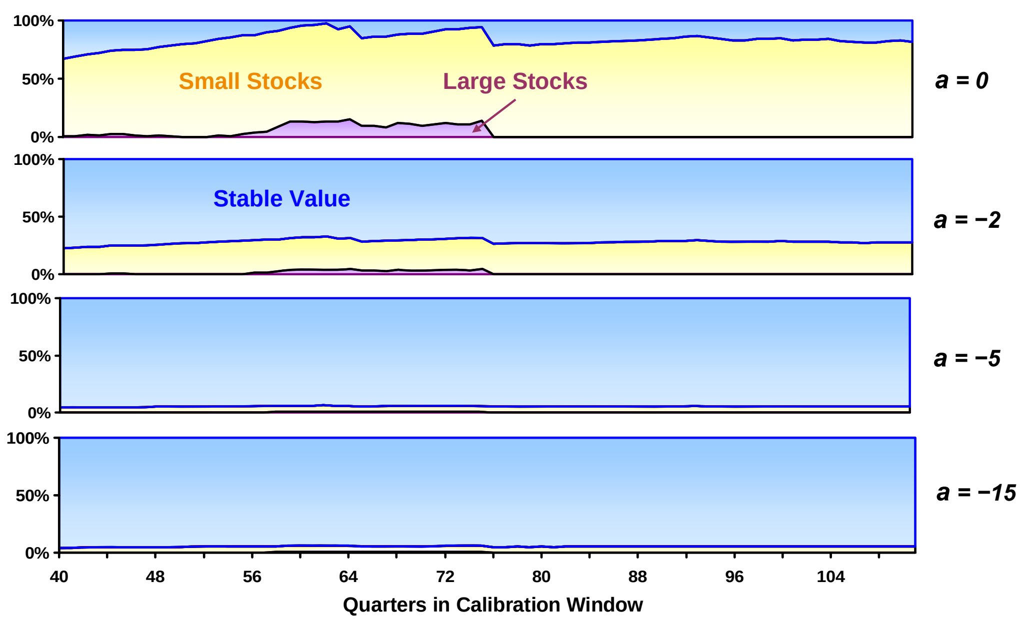


Figure 8. Asset Allocation for Q4-2006 across Four Levels of Riska Aversion (Large Stock Premium = 2.5%)



For any of the risk parameters used, there is a fairly constant allocation of wealth in the Q4-2006 optimal portfolio, irrespective of the number of data quarters used in the calculation. Long-term government and corporate bonds, intermediate government bonds and money market instruments are not included in the portfolio. Interestingly, large stocks are mostly excluded as well, with the exception of calibration windows including the late eighties and early nineties (windows with 56 through 76 quarters of data).

Concluding Remarks

In this paper we use mean-variance, stochastic dominance and intertemporal optimization analyses to explore the performance of SV funds vis-à-vis U.S. large and small stocks, long-term government and corporate bonds, intermediate government bonds and money market funds, during the period January 1989 through December 2006. Despite the different focus of the three methodologies used, the results we obtain under each analysis reinforce each other in the sense that SV funds outperform the alternative investments we consider in one or more dimensions.

In the mean-variance sense, including SV funds in efficient portfolios considerably increases expected net return, and SV even predominates over long-term bonds, for levels of risk in the lower half of the observed monthly return volatility range of 0.12% (for SV funds) to 5.53% (for small stocks). In general, if the historical returns and volatility can serve as a proxy for future expectations, efficient portfolios would not include large stocks, long-term corporate bonds, intermediate government bonds, or money market investments. Rather, efficient portfolios would be comprised of long-term government bonds, small stocks, and SV in various proportions that depend on investor risk tolerance.

Stochastic dominance (SD) analysis provides preference orderings among competing investment alternatives for all investors within a certain class of utility functions. The strength of SD analysis resides in the minimal requirements imposed on preferences but at the same time this may result in an inability to rank alternative investments that could be more easily ranked according to more restrictive approaches such as mean-variance. It is therefore quite remarkable to have found that SV investments stochastically dominate money market funds by the first and higher degrees. They also dominate intermediate government funds by the second and higher degrees, including dominance for the important class of investors characterized by DARA preferences. No other investment alternative was found to dominate SV (or any other) funds.

Intertemporal optimization methods allow us to use the full joint empirical return distribution of alternative investments in order to determine optimal wealth allocations that depend on the risk aversion parameter of the investor. This analysis concludes that, for moderately risk-averse investors, SV funds are a predominant component of an optimal portfolio, to the exclusion of money market funds and near exclusion of intermediate government bonds, and long-term corporate and government bonds.

While important, the implications of our three-fold analysis should not be regarded as dispositive. It should be recalled that over the 18-year period of our analysis, which began with the inception of stable value funds, interest rates exhibited a general decline to half their initial level, albeit with occasional and protracted periods of rising interest rates interspersed. Such a period of decline would tend to favor longer duration fixed income investments, including stable value, over money market funds. We sought to remedy this by examining the precursors of stable value funds – traditional GICs – to see how they fared during the period of rapidly rising interest rates that characterized the late 1970s and early 1980s. What we found was that the GIC index fared well under all three methods of analysis that were used in the present study. However, there were some differences. For example, the GIC returns exhibited second-, third- and fourth-degree stochastic dominance over money market funds, but no longer maintained first-degree dominance. This is because during periods of rapidly rising interest rates money market returns can exceed stable value returns for brief periods, and during the earliest years of the extended period (28 years) of analysis money market rates soared higher than GIC yields did.

We also noted that stable value funds no longer achieved second-, third-, and fourth-degree stochastic dominance over intermediate government bonds, although they almost achieved dominance in each degree, failing to dominate because of exceptionally high bond returns in only three of the 348 months included in our analysis.

Finally, the remarkable performance of stable value funds was achieved, in part, by taking advantage of the term premium that has existed traditionally in the fixed income market. Going forward, there is no guarantee that the term premium will remain as positive as it has been.

REFERENCES

Arditti, Fred "Risk and the Required Return on Equity." *The Journal of Finance* 22:1, 19-36 (March 1967).

Bogle, John C. "The Relentless Rules of Humble Arithmetic." *Financial Analysts Journal* 61:6, 22-35 (November/December 2005).

Cvitanic, Jaksa, Vassilis Polimenis, and Fernando Zapatero "Optimal portfolio allocation with higher moments." forthcoming in *Annals of Finance* (2007).

Dear, Roger and Drezner, Zvi "On the Significance Level of the Multirelation Coefficient." *Journal of Applied Mathematics & Decision Sciences* 1:2, 119-130 (1997).

Dichev, Ilia D. "What Are Stock Investors' Actual Historical Returns? Evidence from Dollar-Weighted Returns." *American Economic Review* 97:1, 386-401 (March 2007).

Donahue, Paul J. "What AICPA SOP 94-4 Hath Wrought: The Demand Characteristics, Accounting Foundation and Management of Stable Value Funds." *Benefits Quarterly* 16:1 (1st Quarter 2000).

Douglas, G. W. "Risk in the Equity Markets: An Empirical Appraisal of Market Efficiency." *Yale Economic Essays* 9, 3-45 (1969).

Drezner, Zvi "Multirelation — a correlation among more than two variables." *Computational Statistics & Data Analysis* 19:3, 283-292 (Fall 1995).

Fabozzi, Frank J., ed. *The Handbook of Stable Value Investments.* Frank J. Fabozzi Associates, New Hope, Pennsylvania (1998).

Grauer, Robert R. and Nils H. Hakansson "Higher Return, Lower Risk: Historical Returns on Long-Run, Actively Managed Portfolios of Stocks, Bonds and Bills, 1936-1978." *Financial Analysts Journal* 38:2, 39-53 (March/April 1982).

Grauer, Robert R. and Nils H. Hakansson "Returns on Levered, Actively Managed Long-

Run Portfolios of Stocks, Bonds and Bills, 1934-1983." *Financial Analysts Journal* 41:5, 24-43 (September/October 1985).

Grauer, Robert R. and Nils H. Hakansson "A Half Century of Returns on Levered and Unlevered Portfolio of Stocks, Bonds, and Bills, with and without Small Stocks." *Journal of Business* 59:2, 287-318 (April 1986).

Grauer, Robert R. and Nils H. Hakansson "Gains from International Diversification: 1968-85 Returns on Portfolios of Stocks and Bonds." *The Journal of Finance* 42:3, 721-739 (July 1987).

Grauer, Robert R. and Nils H. Hakansson "On the Use of Mean-variance and Quadratic Approximations in Implementing Dynamic Investment Strategies: a Comparison of Returns and Investment Policies." *Management Science* 39:7, 856-871 (July 1993).

Grauer, Robert R. and Nils H. Hakansson "Gains From Diversifying Into Real Estate: Three Decades of Portfolio Returns Based on the Dynamic Investment Model." *Real Estate Economics* 23:2, 117-159 (Summer 1995).

Hadar, J. and W. R. Russell "Rules for Ordering Uncertain Prospects." *American Economic Review* 59, 25-34 (1969).

Hanoch, G. and Haim Levy "The Efficiency Analysis of Choices Involving Risk." *Review of Economic Studies* 36, 335-346 (1969)

Hakansson, Nils "On Optimal Myopic Portfolio Policies, With and Without Serial Correlation of Yields." *Journal of Business* 44:3, 324-334 (July 1971).

Hakansson, Nils "Convergence to Isoelastic Utility and Policy in Multiperiod Portfolio choice." *Journal of Financial Economics* 1:3, 201-224 (September 1974).

Huberman, Gur and Stephen Ross, "Portfolio Turnpike Theorems, Risk Aversion and Regularly Varying Utility Functions." *Econometrica* 51:3, 1104-1119 (September 1983).

Jacquier, Eric, Alex Kane, and Alan J. Marcus "Optimal Estimation of the Risk Premium for the Long Run and Asset Allocation: A Case of Compounded Estimation Risk." *Journal of Financial Econometrics* 3:1, 37-55 (Winter 2005).

Leland, Hayne "On Turnpike Portfolios." In *Mathematical Methods in Investment and Finance,* Karl Shell, G. P. Szego, eds., Amsterdam, North-Holland, 1972.

Levy, Haim *Stochastic Dominance: Investment Decision Making Under Uncertainty, 2nd Edition.* Springer, 2006.

Markowitz, Harry M. *Mean-Variance Analysis in Portfolio Choice and Capital Markets,* Oxford UK: Basil Blackwell, 1987.

Merton, Robert C. "On Estimating the Expected Return on the Market: An Exploratory Investigation." *Journal of Financial Economics* 8:4, 323-361 (December1980).

Mossin, Jan "Optimal Multiperiod Portfolio Policies." *Journal of Business* 41:2, 215-229 (April 1968).

Paradis, Vicoria M. "Challenges of Measuring Performance for Stable Value Funds." *Journal of Performance Measurement* 5:3 (Spring 2001).

Polimenis, Vassilis "Skewness Correction for Asset Pricing." Working Paper, Anderson Graduate School of Management, University of California, Riverside (April 2006).

Ross, Stephen "Portfolio Turnpike Theorems for Constant Policies." *Journal of Financial Economics* 1:2, 171-198.

Ryan, Ronald J. "GIC Indices and Their Risk/Return Behavior." *The Handbook of Stable Value Investments* 11-16 (1998).

Siegel, Jeremy J. "Perspectives on the Equity Risk Premium." *Financial Analysts Journal* 61:6, 61-73 (November/December 2005).

Stable Value Investment Association (SVIA), Response to FASB December 9, 2004 Questions Regarding Commingled Stable Value Funds, January 7, 2005.

Tobe, Christopher B. "The Consultant's Guide to Stable Value." *The Journal of Investment Consulting* 7:1 (Summer 2004).

Appendix A. An Alternative Stochastic Dominance Algorithm

In this appendix we present a recursive algorithm that can be easily applied to test stochastic dominance of an arbitrary degree (up to fourth degree in most practical applications).

The theory and practice of stochastic dominance (SD) is discussed in great detail in Levy (2006). In Chapter 5, Levy describes algorithms used to test for first, second, and third degree stochastic dominance (FSD, SSD and TSD, respectively). The inputs for these algorithms are the sorted realized rates of return, for the period of interest, of the assets for which SD is to be tested. For a given asset, the sorted rates of return can be used to construct an empirical cumulative distribution function (CDF), assuming that each return has a probability $1 \div n$ of occurring, where n is the number of observations and where repeated returns have probability $h \div n$, and h is the number of repeated values.

Levy (2006) presents separate algorithms for FSD, SSD and TSD. The algorithms for FSD and SSD are based on pairwise comparison of sorted returns and cumulative sorted returns, respectively, for the two asset classes being checked.

For FSD, this approach amounts to computing the difference of the empirical CDFs of the two asset classes for each point $i \div n$, with $i = 1, 2, \ldots, n$. For SSD, the approach amounts to computing the integral of the CDFs difference, again for each point $i \div n$, with $i = 1, 2, \ldots, n$. Note that the difference calculation for FSD and the integral of these differences for SSD are in effect calculated over the y-axis. Because of the linear functions involved, this approach yields the same test results as if the calculations had been done over the horizontal axis.

In the case of third or higher degree SD, however, this equivalence no longer holds, since the integrals now become non-linear functions of the values in the respective axis and, furthermore, using the same approach as in the FSD or SSD algorithms may lead to wrong conclusions. For this reason, Levy (2006, Section 5.4) presents algorithms for FSD, SSD and TSD based on integrals of the CDFs of the two asset classes. However, since the set of realized returns is different for each investment, these algorithms first have to be applied separately to the individual CDF of each investment and then combined in order to compare the resulting integrals at each point of interest.

We use an alternative approach that allows the derivation of a simple, unified and recursive algorithm, directly applied to the difference between the CDFs of the two asset classes being compared and having the same structure, independent of the degree of stochastic dominance for which one wants to test.

Our algorithm can be derived by (1) considering the union of realized returns for both asset classes over the period of interest as the set of all possible values under the distribution of a given asset class and (2) assigning zero probability to those return values that are not realizations of that asset class. For realized values, the empirical probability of a return value is taken to be the relative frequency with which that value is observed, i.e. $h \div n$ for values that are repeated h times in a sample of n observations. We let m denote the number of unique realized returns across both asset classes.[17]

Table A-1 illustrates how hypothetical realized return data on two investments, F and G, observed over 20 periods are used to construct the set of unique realized returns and their empirical probability density functions, $f(r)$ and $g(r)$.

The first panel of Table A-1 shows the sorted hypothetical returns for the two investments. The second panel shows their empirical probability density functions (pdf). For instance, a –4% return has frequency 20% for the first asset and 0% for the second.

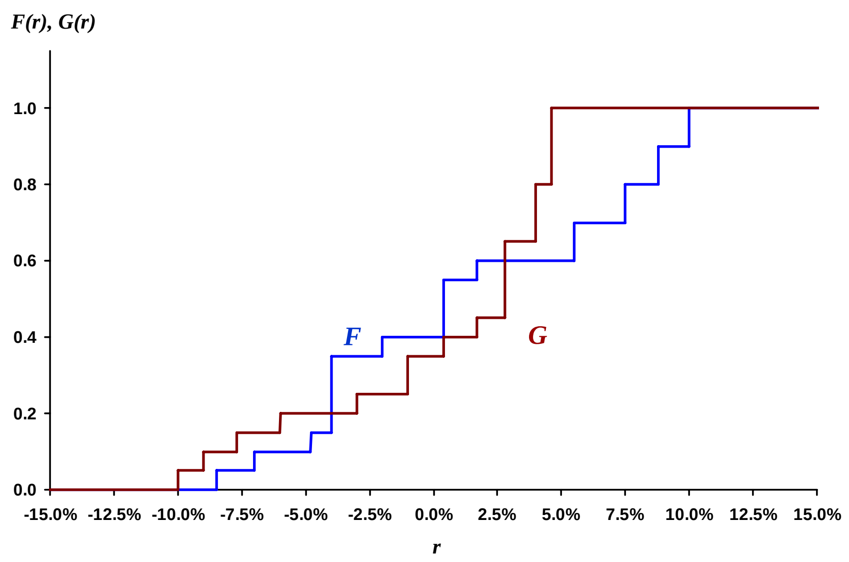
Figure A-1 plots the empirical cumulative distribution functions (CDF) of the two investments.

[17] For a large sample of realized returns, m will generally be greater than the sample size n, but there is no theoretical reason why this should be the case.

Table A-1. Distribution of Returns for Two Asset Classes

	Sorted Returns		Empirical pdf of Union of Returns		
Obs.	*F*	*G*	Return, *r*	*f*(*r*)	*g*(*r*)
1	-8.5%	-10.0%	-10.0%	0.00	0.05
2	-7.0%	-9.0%	-9.0%	0.00	0.05
3	-4.8%	-7.7%	-8.5%	0.05	0.00
4	-4.0%	-6.0%	-7.7%	0.00	0.05
5	-4.0%	-3.0%	-7.0%	0.05	0.00
6	-4.0%	-1.0%	-6.0%	0.00	0.05
7	-4.0%	-1.0%	-4.8%	0.05	0.00
8	-2.0%	0.4%	-4.0%	0.20	0.00
9	0.4%	1.7%	-3.0%	0.00	0.05
10	0.4%	2.8%	-2.0%	0.05	0.00
11	0.4%	2.8%	-1.0%	0.00	0.10
12	1.7%	2.8%	0.4%	0.15	0.05
13	5.5%	2.8%	1.7%	0.05	0.05
14	5.5%	4.0%	2.8%	0.00	0.20
15	7.5%	4.0%	4.0%	0.00	0.15
16	7.5%	4.0%	4.6%	0.00	0.20
17	8.8%	4.6%	5.5%	0.10	0.00
18	8.8%	4.6%	7.5%	0.10	0.00
19	10.0%	4.6%	8.8%	0.10	0.00
20	10.0%	4.6%	10.0%	0.10	0.00

Figure A-1. Empirical CDFs of Two Asset Classes



Visual inspection of Figure A-1 reveals that no return distribution exhibits FSD over the other since the vertical distance $G(r) - F(r)$ changes sign for different values of r. More precisely, define

$$I_1(r) = G(r) - F(r) = G(r_i) - F(r_i) \equiv p_i, \quad r_i \le r < r_{i+1}, \quad i = 1, \cdots, m. \qquad \text{(A1)}$$

In particular, we want to evaluate $I_1(r)$ at the joint unique realized return values, $r_1,\ r_2,\ \cdots,\ r_m$. The definition for theoretical k^{th}-degree stochastic dominance requires

$$I_k(r) = \int_{-\infty}^{r} I_{k-1}(z)dz \ge 0, \ \text{ with } I_1 \text{ given by (1) above, } k = 1,\ 2, \cdots, \qquad \text{(A2a)}$$

$$I_j(r_m) \ge 0, j = 2,\ 3, \cdots,\ k-1, \qquad \text{(A2b)}$$

for all r, with at least one strict inequality.

The arrangement of the observed returns for investments F and G illustrated in Table A-1 allows us to calculate the integrals in (A2a) and (A2b) directly from the data. Specifically,

$$I_2(r) = I_2(r_{i-1}) + I_1(r_{i-1})(r - r_{i-1}), \quad \text{for} \quad r_{i-1} \le r < r_i, \quad i = 2,\ \cdots,\ m, \quad \text{with} \quad I_2(r_1) = 0. \qquad \text{(A3)}$$

This expression can be easily evaluated at the observed return values $r_1,\ r_2,\ \cdots,\ r_m$. In general, and corresponding exactly to the theoretical conditions in (A2a) and (A2b), it can be shown that

$$I_k(r) = \sum_{j=0}^{k-1} \frac{1}{j!} I_{k-j}(r_{i-1})(r - r_{i-1})^j, \quad \text{for} \quad r_{i-1} \le r < r_i, \qquad \text{(A4)}$$

with $I_k(r_1) = 0, \quad i = 2, 3, \cdots, m; \quad k = 2, 3, \cdots$.

Note that the expression given in (A4) allows us to recursively calculate the test integrals for any desired degree of stochastic dominance at points on interest, once the CDF difference in (A1), based on the union of unique realized returns for the two asset classes of interest, is obtained. The conditions in (A2a) and (A2b) can then be checked in order to determine stochastic dominance.

A conclusive determination of third- and higher-degree SD, however, cannot be based on checking conditions (A2a) and (A2b) at the observed return points alone. The reason is that the integrals $I_3(r)$, $I_4(r)$, … are non-linear functions of r within a given interval $[r_{i-1}, r_i)$. We next discuss the additional conditions for interior points in the context of our example.

Table A-2 shows the results of the dominance tests (up to fourth degree) applied to the hypothetical return data of Table A-1.

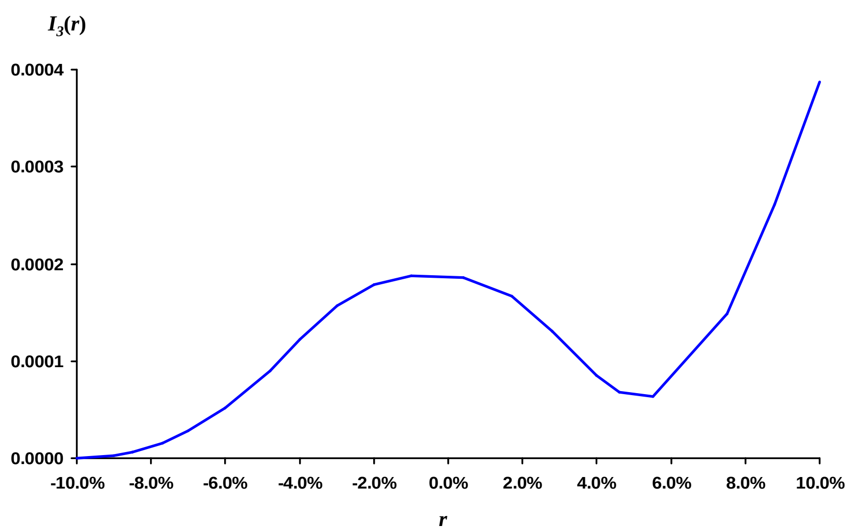
Table A-2. Stochastic Dominance Tests

	pdf		*CDF*		*FSD*	*SSD*	*TSD*	*4SD*
Return, *r*	***f(r)***	***g(r)***	***F***	***G***	I_1	I_2	I_3	I_4
-10.0%	0.00	0.05	0.00	0.05	0.05	0.0000	0.000000	0.000000000
-9.0%	0.00	0.05	0.00	0.10	0.10	0.0005	0.000003	0.000000008
-8.5%	0.05	0.00	0.05	0.10	0.05	0.0010	0.000006	0.000000029
-7.7%	0.00	0.05	0.05	0.15	0.10	0.0014	0.000016	0.000000115
-7.0%	0.05	0.00	0.10	0.15	0.05	0.0021	0.000028	0.000000266
-6.0%	0.00	0.05	0.10	0.20	0.10	0.0026	0.000052	0.000000661
-4.8%	0.05	0.00	0.15	0.20	0.05	0.0038	0.000090	0.000001496
-4.0%	0.20	0.00	0.35	0.20	-0.15	0.0042	0.000122	0.000002342
-3.0%	0.00	0.05	0.35	0.25	-0.10	0.0027	0.000157	0.000003747
-2.0%	0.05	0.00	0.40	0.25	-0.15	0.0017	0.000179	0.000005430
-1.0%	0.00	0.10	0.40	0.35	-0.05	0.0002	0.000188	0.000007275
0.4%	0.15	0.05	0.55	0.40	-0.15	-0.0005	0.000186	0.000009904
1.7%	0.05	0.05	0.60	0.45	-0.15	-0.0025	0.000167	0.000012223
2.8%	0.00	0.20	0.60	0.65	0.05	-0.0041	0.000131	0.000013876
4.0%	0.00	0.15	0.60	0.80	0.20	-0.0035	0.000085	0.000015163
4.6%	0.00	0.20	0.60	1.00	0.40	-0.0023	0.000068	0.000015618
5.5%	0.10	0.00	0.70	1.00	0.30	0.0013	0.000063	0.000016183
7.5%	0.10	0.00	0.80	1.00	0.20	0.0073	0.000149	0.000018107
8.8%	0.10	0.00	0.90	1.00	0.10	0.0099	0.000261	0.000020737
10.0%	0.10	0.00	1.00	1.00	0.00	0.0111	0.000387	0.000024610

Table A-2 shows that $I_1(r)$ (defined here as $G - F$) changes sign and so no distribution stochastically dominates the other in the first-degree sense. Similarly, $I_2(r)$ alternates sign, which means that neither F SSD G nor G SSD F.

By contrast, we see that $I_3(r)$ is positive for all relevant points and $I_2(10.0\%) > 0$. This allows us to conclude that F TSD G. Figure A-2 plots the function $I_3(r)$ for the relevant range of observed returns in the example of Table A-1.

Figure A-2. Third Degree Stochastic Dominance Test



As mentioned above, since the $I_3(r)$ function is non-linear between any two contiguous observed return points, one should also check interior points other than the observed returns in Table A-2. As Levy (2006) points out, this needs only be checked at interior points between two observed return points for which $I_3(r)$ turns from a decreasing function to an increasing function or, equivalently, when $I_2(r)$, which is the first derivative of $I_3(r)$, changes from negative to positive (between 4.6% and 5.5%, as shown in Table A-2).

With no loss of generality, let the two contiguous observed return points for which $I_3(r)$ turns from a decreasing function to an increasing function be r_{i-1} and r_i and note that the function $I_3(r)$ is a second degree polynomial over $[r_{i-1}, r_i)$:

$$y = \alpha + \beta(r - r_{i-1}) + \chi(r - r_{i-1})^2 \quad \text{(A5)}$$

with $\alpha = I_3(r_{i-1})$, $\beta = I_2(r_{i-1})$, and $\chi = \frac{1}{2} I_1(r_{i-1})$ being known constants. The minimum value of expression (A5) is attained at

$$r_{\min} = r_{i-1} - \frac{\beta}{2\chi} = r_{i-1} - \frac{I_2(r_{i-1})}{I_1(r_{i-1})}. \tag{A6}$$

For our example, this value is r_{min}=5.175% with $I_3(r_{min})$=0.000061. Together with the information on Table 2, the positive value of $I_3(r_{min})$, leads to the conclusion that *F* TSD *G*. From this result, it follows as a corollary that *F* also dominates *G* stochastically for any degree higher than third.[18]

[18] Levy (2006) has a discussion on how existing TSD algorithms fail to check for interior points, but errors crept in apparently during the editorial process such that his example in Section 5.5, pp. 189 and 190, appears to be flawed. First, there seems to be missing data for an additional period in Levy's table on p. 189. This missing period needs to have observed returns of 10% for distribution *F* and 5% for distribution *G* in order to be able to exactly match the figures reported by Levy on p. 189. But even when this is taken care of, and Levy's calculations are reproduced, it is the case that the interior point he correctly identifies, at a return of 25%, gives a theoretical minimum value for I_3(25%) of exactly zero. What Levy considers a negative minimum value of $-3.3\text{x}10^{-7}$ (incorrectly reported as positive $3.3\text{x}10^{-7}$ in Levy, p. 189) is simply due to rounding error and so his conclusion that "… *F* does not dominate *G* by the TSD …" is not valid. Indeed, for Levy's example, straightforward application of the compact formulae and interior points check developed in this paper would conclude that *F* dominates *G* by the TSD.