

Liquidity and Valuation in an Uncertain World

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ABSTRACT

In the current credit crisis there is little or no trade in a variety of financial assets, even though bids and asks exist for many of these assets. We develop a model in which this illiquidity arises from uncertainty, and we argue that this new form of illiquidity makes bid and ask prices unsuitable as metrics for establishing "fair value" for these assets. We show how the extreme uncertainty that traders currently face can be characterized by incomplete preferences over portfolios, and we use Bewley's [2002] model of decision making under uncertainty to derive equilibrium quotes and the non-existence of trade at these quotes. Having established the origin of the quotes, and why the market freezes, we are then able to use our approach to suggest alternatives for valuing assets in illiquid markets.

KEYWORDS: liquidity, uncertainty, subprime crisis, fair value accounting

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1. Introduction

A puzzling feature of the current credit crisis is the lack of trading volume in many affected markets. While dealers continue to post bid and ask prices for mortgage-backed securities and CLO tranches, there is virtually no trading at any of these prices. The auction-rate preferred market has faltered as new buyers of paper have departed the scene, leaving holders of such paper unable to follow suit due to the lack of trading in the market. From leveraged loans to residential real estate, a dearth of actual trading has resulted in markets not so much failing as “freezing”, with traders apparently unwilling to either buy or sell at almost any posted price. Such a scenario is hard to reconcile with our standard models of asset price formation in which supply and demand curves inevitably find an equilibrium price, or even with microstructure models that allow for bid and ask prices to evolve separately.

Two issues naturally come to the fore. First, what is causing such seemingly aberrant market behavior? And, second, what are these financial assets worth given that there is no actual trading? Addressing the first issue requires understanding how, for a given financial asset, an equilibrium can emerge with a multiplicity of prices at which no, or very limited, trade actually occurs. Such an outcome is inconsistent with the typical view in financial markets that at a given price “if you are not a buyer then you are a seller”, but instead reflects a reality in which traders apparently will neither buy nor sell even at drastically different prices. Understanding asset valuation in such a world then takes on particular importance because, with no transactions, the only prices are bid or ask prices at which no one actually transacts. Which price, if any, reflects

a “fair value” is far from obvious, requiring firms, and their accountants, to grapple with difficult valuation issues.

In this research, we suggest answers to these questions in the context of a model in which agents have incomplete preferences over portfolios. That is, we model traders who have to make decisions during the current credit crisis as if they simply cannot rank order some portfolios. This is a reflection of the extreme uncertainty that they face. For some portfolios, traders know how they feel and can say that one portfolio is better than another; for other portfolios, for example, those with exposure to credit derivatives, they are uncertain about how to rank them. Although most analyses of asset markets (and for that matter just about everything else) in finance or economics begin with complete preferences, or representations that reflect complete preferences, this is typically driven more by convenience than by economic reality.¹ The assumption of complete preferences is neither a natural nor we would argue a realistic assumption to use in attempting to understand the current credit crisis

Our analysis draws on the original insight of Knight [1921] that uncertainty can play an important, and distinct, role in influencing agents’ behavior in risky markets. One approach to modeling Knightian uncertainty is due to Schmeidler [1989] and Gilboa and Schmeidler [1989] who weaken the independence axiom of Savage [1952] and show that this leads to a representation with a single utility function and set of beliefs which can be interpreted as a reflection of the ambiguity in probabilities that decision makers face. Numerous authors have shown that uncertainty, or ambiguity as it is also called, when modeled in this way can induce

¹ The completeness assumption that underlies expected utility analysis requires that the decision maker whose preferences are to be represented can rank order every pair of alternatives. If one restricts the set of alternatives to ones that the decision maker has experience with, and considers a world in which that experience is clearly relevant, this assumption is not too obnoxious. But even in this restrictive setting if the set of alternatives is rich enough the completeness assumption when combined with transitivity often leads to contradictions. For a discussion of this point see Kreps [1988]. In the setting we consider in this paper traders are faced with an environment that was unexpected and with which they have no experience. In this setting completeness is surely too strong.

non-participation in markets, which is certainly consistent with some aspects of the current market crisis. However, if traders have Gilboa-Schmeidler style preferences and enter the period of the credit crisis with a portfolio containing the now ambiguous financial assets, they will be inclined to leave the market by selling their position.² But a desire to leave a market would also generate trading volume as current holders move away from certain assets, a result seemingly inconsistent with the absence of trading in the current crisis.

Using Bewley's [2002] model of Knightian uncertainty, we show that this absence of trading naturally arises when traders have incomplete preferences over portfolios.³ Bewley's approach applied to preferences over portfolios yields a representation consisting of a set of beliefs and a utility function for wealth. In his representation, one portfolio is preferred to another if and only if it yields greater expected utility for every belief in the set of beliefs that represent the trader's preferences. Of course, this approach by itself does not result in a complete model of decisions as a trader with incomplete preferences may be unable to decide. Bewley supplements his representation with an "inertia" assumption to provide a complete theory. A trader moves away from the status quo if and only if the move (a trade) is expected utility improving for every belief in the set of beliefs that represent the trader's preferences. This "inertia" assumption results in the status quo prevailing in some circumstances, and sets the stage for our model of why asset markets can appear to freeze in the face of uncertainty.

² Ambiguity aversion is often interpreted as generating no-trade results. Whether this is true or not depends on the trader's endowments. Consider a two-trader pure exchange economy with no aggregate risk in which one trader is risk averse and Gilboa-Schmeidler style ambiguity averse, the other trader is a standard expected utility maximizer, and the beliefs of the EU trader are in the set of beliefs used to represent the ambiguity averse trader's preferences. There is a kink in the indifference curves of the ambiguity averse trader along the certainty line, so for a range of prices determined by his risk aversion and set of beliefs this trader will demand a certainty allocation. If initial endowments are on the diagonal of the Edgeworth box, then there will be no trade. However, if initial endowments were off the diagonal, then there would be trade to the diagonal. So these results should not be interpreted as no-trade results, but rather as results about what equilibrium allocations have to look like.

³ Bewley's work was originally distributed as Cowles Foundation Working paper no. 807 in 1986. Aumann [1962] provided an earlier representation theorem for incomplete preferences, but his approach is not tailored to the question posed by Knight.

A simple example can explain our intuition. Suppose an investor is holding a CLO and is given a bid price, or mark, by a dealer of 57. A price of 57 may strike the investor as far too low, a product perhaps of over-reaction on the part of the dealer. If the trader believed the CLO could be worth 75, then surely he would not sell the asset at 57. Should he instead buy more of the asset? Now the problem is that the dealer may be underestimating the extent of the CLO problem, and in a meltdown the CLO could be worth only 35. If this is the case, then surely he should not buy at 57. Because the trader is uncertain about the CLO market, he does not have a single prior belief on the CLO value and instead evaluates each possible trade relative to his set of possible beliefs. In the presence of this uncertainty, the investor neither buys nor sells, and the market falters.

There are, of course, other, more standard explanations that can be applied to the current credit crisis. Two of the most likely candidates are a change in the market price of risk, and asymmetric information. Because our model allows for both risk and uncertainty to affect traders' demands, we can demonstrate the specific roles played by each in affecting equilibrium. While an increase in the price of risk will cause asset prices to fall, it also engenders substantial trade. By contrast, the addition of uncertainty quells trading, and, perhaps more importantly, changes the entire price-setting process. In particular, unlike in standard economic models where the single equilibrium price is an "average" across beliefs, in our uncertainty world there are a range of market prices. Each of these prices reflects individual beliefs about possible outcomes, and a "spread" arises between these optimistic and pessimistic-based prices.

Microstructure models also feature a set of prices, and a spread arises in such models to reflect the risks of trading with individuals who have superior information. The "uncertainty spread" in our model differs in important ways from this microstructure spread, and reflects the

fact that illiquidity arises in our world from the uncertainty and not from risk. As we demonstrate, this distinction has important implication for the properties of prices and the nature of trading in our markets.

Our analysis of price-setting in an uncertain world has interesting implications for the debate regarding establishing “fair values” for assets trading in illiquid markets. While the Financial Accounting Standards Board has decreed that the fair value is “the price that would be received to sell an asset or transfer a liability in an orderly transaction between market participants at the measurement date”, implementing this in current markets has proven problematic⁴. Our model with uncertainty demonstrates that while quoted prices will exist, these prices have biases relative to the prices that would prevail in “normal” markets without uncertainty. These biases arise because in the presence of uncertainty prices are not averages across possible outcomes or even across individuals, but rather are individual beliefs about the best and worst case outcomes. We argue that this renders bid and ask prices unsuitable as metrics for fair value, and we suggest alternatives for valuing assets in markets characterized by uncertainty as well as risk.

This paper is organized as follows. In the next section, we develop a model of trading in which traders have incomplete preferences and evaluate portfolios in the spirit of Bewley [2002]. We derive traders’ equilibrium demands, and investigate how prices change and what trades occur if there is an unanticipated shock to the expected future value of the stock. We derive our results when traders agree on the size of the shock, and when they face uncertainty as to its size. We demonstrate conditions under conditions under which the ambiguity results in a no-trade

⁴ In September 2006, the Financial Accounting Standards Board [FASB] issued SFAS 157 “Fair Value Measurements” (denoted “FAS 157”) which provides guidance to companies on how entities should measure fair value for financial reporting requirements. In February 2007, the FASB issued a related standard SFAS 159 “The Fair Value Option for Financial Assets and Financial Liabilities (denoted “FAS 159”) which provides guidance to firms on what can be fair valued and the disclosure requirements for doing so.

equilibrium. In Section 3, we then consider how to attach “fair values” to assets when uncertainty induces illiquidity. We investigate alternative valuation metrics, and propose alternatives for establishing fair values. The paper’s final section is a conclusion.

2. The Model

Trade in our economy takes place at two dates, $t = 0, 1$. At time 0, traders trade a risky asset and a risk-free asset.⁵ At time 1, an unanticipated shock to trader’s beliefs about the future value of the risky asset occurs and traders can re-trade. After period 1 ends, asset payoffs are realized.

The risk free asset, cash, has a constant value of 1. The risky asset has a price of p_t per unit at date $t = 0, 1$ and an uncertain future value, to be realized at the end of period 1, denoted \tilde{v} . We will interpret the risky asset as CDOs or MBSs, but our analysis would also apply to other risky asset such as bonds, stocks or houses.

There are I traders indexed by $i = 1, \dots, I$. These traders have heterogeneous beliefs about the future value of the risky asset. They all believe that the future value is normally distributed with variance σ^2 , but they disagree about the mean.⁶ Trader i ’s mean is \bar{v}_i and we assume that for at least two traders i and j , $\bar{v}_j \neq \bar{v}_i$. These heterogeneous beliefs are not the result of traders with a common prior having received differential information; so traders do not attempt to infer each other’s beliefs, and thus information, from prices. Instead, these traders simply disagree about the future value of the risky asset.

⁵ We focus on the case of one risky asset in order to simplify the analysis. It would also be interesting to consider many risky assets with a factor structure and ambiguity about a factor.

⁶ We could also, or instead, have heterogeneous beliefs about the variance of the future value of the asset. This would not affect our qualitative results.

We believe that heterogeneous beliefs are natural and this assumption is crucial for the existence of trade. If traders have common beliefs then standard no-trade theorems (see Milgrom and Stokey [1982]) would apply. Although there may be trade in an initial period, once an optimal allocation is reached, the arrival of new information will not generate trade.⁷ We do not believe that this phenomenon is the reason for the lack of trade in the current credit crisis; instead, it illustrates the inadequacy of Walrasian equilibrium models with homogeneous beliefs and complete markets as a tool to use in understanding trade and liquidity in asset markets.

Trader i 's endowment of the risky asset is \bar{x}_i . The per capita endowment of the risky asset is thus $\bar{x} = \sum_{i=1}^I \bar{x}_i$. We do not give the traders endowments of cash, but none of our results depend on this simplifying assumption. All traders have constant absolute risk aversion (CARA) utility of wealth, w , at the end of period one. To keep the notation as clean as possible we assume that they have a common risk aversion coefficient of 1.

At time 0, each trader simply maximizes expected utility of wealth w given beliefs and the time zero price, p_0 , of the risky asset. Our traders are unaware of the possibility of a shock at time 1, and so do not plan for it. Thus the shock can be viewed as an 'unforeseen contingency'; an event that the traders do not think about when constructing their portfolios as they do not know that it is possible.⁸ At time 1, the unanticipated shock occurs. To make the interpretations of our results consistent with the recent liquidity crisis we model the shock as a decline in the expected future value of the asset, but our formal analysis is consistent with any shock.

⁷ This claim depends on the traders having access to effectively complete markets. If instead markets were to be incomplete, then the initial round of trade need not lead to Pareto optimal allocations and the arrival of information can lead to re-trade, see Blume, Coury and Easley [2006].

⁸ The entry in the Palgrave by Lipman [2008] provides a discussion of the recent literature on modeling unforeseen contingencies.

We investigate two scenarios for how beliefs change in response to the shock. First, to establish a benchmark case, we ask how prices change and what volume of trade occurs if there is a common percentage decline in the expected future value of the risky asset. That is, in this scenario, trader i 's expected future value of the risky asset declines to $\bar{v}_{i1} = \alpha \bar{v}_i$, where $\alpha < 1$.⁹ Such a scenario is consistent with an overall change in the market price of risk, and it corresponds to a more standard analysis in which traders evaluate portfolios only on expected returns and risk. Second, we ask about prices and the volume of trade if the shock is ambiguous. In this scenario, all traders know that the future value of the asset has declined, but they do not know the magnitude of the decline. After the occurrence of the period one shock our traders do not have induced preferences over portfolios that have a Savage (1952) subjective expected utility representation. So they do not act as subjective expected utility maximizers who can access a distribution over the set of possible declines in asset value and then act according to the predicted distribution. Instead, we view them as having incomplete preferences over portfolios. After the shock, they view the world as ambiguous and they simply cannot rank some alternatives.

We model the scenario two decision problem using Bewley's [2002] approach to Knightian decision theory. Our traders each have a set of beliefs about the future value of the risky asset and they trade away from the status quo, their current portfolio, only if the trade is beneficial according to every belief they consider. Bewley shows that under standard assumptions incomplete preferences can be represented with a single utility function and a set of probabilities. With this representation, one act is preferred to another act if and only if it yields greater expected utility for every probability in the set. Some pairs of acts are not ranked, so to

⁹ Our analysis could be done for any α . We focus on the case of a fall in asset values to describe the current credit crisis.

produce a theory of action additional structure is needed. Bewley does this with his Inertia Assumption that the decision maker moves away from the status quo only if the move increases expected utility for every probability in the decision maker's set of probabilities. This structure induces an endowment effect or a status quo bias similar to, but different in motivation from, the effect first studied by Thaler [1980].

We follow this approach in scenario two by assuming that the set of possible declines in the expected value of the risky asset is described by $1 - \alpha$ where $\alpha \in [\underline{\alpha}, \bar{\alpha}]$ with $\bar{\alpha} < 1$. We define the status quo for a trader as the portfolio the trader brings into period 1.

2.1 Period 0 Equilibrium

In period 0, traders choose portfolios to maximize the expected utility of wealth using their initial beliefs. If trader i chooses the portfolio (x_i, m_i) his future wealth (after period 1) will be the random variable $\tilde{w}_i = \tilde{v}x_i + m_i$. He has CARA utility and the distribution of wealth is normal so his expected utility from this random wealth is given by a monotonic transformation of the mean-variance expression

$$(\bar{v}_i - p_0)x_i - (1/2)\sigma^2(x_i)^2 + \bar{w}_i \quad (1)$$

where \bar{w}_i is the trader's initial wealth. Thus, trader i 's period 0 demand for the risky asset has the standard form

$$x_i^* = \frac{\bar{v}_i - p_0}{\sigma^2}. \quad (2)$$

In equilibrium, per capita demand for the risky asset must equal per capita supply, so

$$1/I \sum_{i=1}^I x_i^* = \bar{x}. \quad (3)$$

The average belief about the future mean value of the risky asset is $\hat{v} = I^{-1} \sum_{i=1}^I \bar{v}_i$. Using this expression for average beliefs, we can write the equilibrium price of the risky asset and equilibrium asset holdings as

$$\begin{aligned} p_0^* &= \hat{v} - \sigma^2 \bar{x} \\ x_{i0}^* &= \left(\frac{\bar{v}_i - \hat{v}}{\sigma^2} \right) + \bar{x}. \end{aligned} \tag{4}$$

Thus, the period 0 price of the risky asset is the average mean future value reduced by a factor that compensates traders for holding the market risk. Traders hold more or less of the market risk according to whether they are optimistic relative to the average or pessimistic relative to the average.

2.2 Period 1 Equilibrium

In scenario one, in which each trader's mean future value of the risky asset declines by α , the calculation of period 1 equilibrium follows the same procedure as we employed to compute the period 0 equilibrium. Let $\hat{v}_1 = (1/I) \sum_{i=1}^I \bar{v}_{i1} = \alpha \hat{v}$ be the mean belief at time one. Then the period 1 equilibrium price and equilibrium asset holdings are given by

$$\begin{aligned} p_1^* &= v_1 - \sigma^2 \bar{x} = \alpha \hat{v} - \sigma^2 \bar{x} \\ x_{i1}^* &= \left(\frac{\bar{v}_{i1} - \hat{v}_1}{\sigma^2} \right) + \bar{x} = \alpha \left(\frac{\bar{v}_i - \hat{v}}{\sigma^2} \right) + \bar{x}. \end{aligned} \tag{5}$$

The optimal trade by trader i is

$$t_i^* = x_{i1}^* - x_{i0}^* = \frac{(\alpha - 1)(\bar{v}_i - v)}{\sigma^2} \tag{6}$$

and aggregate volume is

$$(1/2) \sum_{i=1}^I \left| \frac{(\alpha - 1)(\bar{v}_i - v)}{\sigma^2} \right| > 0. \quad (7)$$

Thus, the price of the risky asset declines and all traders' holdings of the risky asset move toward the average holding, \bar{x} . In this case, the unanticipated shock causes a positive volume of trade as each trader wants to reduce the riskiness of his portfolio.¹⁰

Calculation of the equilibrium under scenario two, in which traders view the magnitude of the decline as ambiguous, does not follow immediately from the period 0 results. Now traders make decisions over trades rather than over final portfolios. In the standard expected utility framework this distinction is irrelevant; with ambiguity it matters. In period one, trader i 's endowment is (x_{i0}^*, m_{i0}^*) which consists of his position in the risky asset and his cash position brought into period one from period zero. Let (t_i, m_i) be trader i 's trade of risky asset and cash. Trader i 's period one budget constraint is then

$$p_1 t_i + m_i = 0. \quad (8)$$

Trader i 's future wealth, given his endowment and trade, is given by

$$\begin{aligned} \tilde{w}_{i1} &= \tilde{v}(x_{i0}^* + t_i) + (m_{i0}^* + m_i) \\ &= \tilde{v}x_{i0}^* + t_i(\tilde{v} - p_1) + m_{i0}^*. \end{aligned} \quad (9)$$

For any expected future value of the risky asset, \bar{v}_{i1} , that trader i considers possible, his expected future wealth is

$$\tilde{w}_{i1} = \bar{v}_{i1}x_{i0}^* + t_i(\bar{v}_{i1} - p_1) + m_{i0}^*. \quad (10)$$

and the variance of his wealth is

¹⁰ Our assumption of heterogeneous beliefs is important for this conclusion. If instead traders had homogeneous prior beliefs the common shock to the future expected value of the asset would not generate trade in period one.

$$\sigma^2(x_{i0} + t_i)^2. \quad (11)$$

Trader i chooses a non-zero trade t_i only if: (i) it is better than the status quo point for all distributions of returns and, (ii) it is not dominated by another trade. Trader i evaluates trades, given a mean, by the mean-variance expression given in equation (1.1). So his trade t_i must solve the following inequalities.

- (i) $\bar{v}_{i1}x_{i0}^* + t_i(\bar{v}_{i1} - p_1) + m_{i0}^* - (1/2)\sigma^2(x_{i0}^* + t_i)^2 \geq \bar{v}_{i1}x_{i0}^* + m_{i0}^* - (1/2)\sigma^2(x_{i0}^*)^2$
for all $\bar{v}_{i1} \in [\underline{\alpha}\bar{v}_i, \bar{\alpha}\bar{v}_i]$, and
- (ii) there does not exist a trade t' such that (12)

$$\bar{v}_{i1}x_{i0}^* + t'(\bar{v}_{i1} - p_1) + m_{i0}^* - (1/2)\sigma^2(x_{i0}^* + t')^2 > \bar{v}_{i1}x_{i0}^* + t_i(\bar{v}_{i1} - p_1) + m_{i0}^* - (1/2)\sigma^2(x_{i0}^* + t_i)^2$$

for all $\bar{v}_{i1} \in [\underline{\alpha}\bar{v}_i, \bar{\alpha}\bar{v}_i]$.

The trader considers his portfolio brought into this period from period 0 as his status quo point and then trades away from it only if the trade is welfare improving for every distribution of returns that he believes to be possible. This results in the trader rejecting trades for a range of prices, and it is this aspect of his trading that we now consider further. Condition (i) implies that there will be no trade by i if

$$\bar{\alpha}\bar{v}_i - \sigma^2x_{i0}^* \geq p_1 \geq \underline{\alpha}\bar{v}_i - \sigma^2x_{i0}^* \quad (13)$$

For prices outside of this no-trade region, trader i has a set of possible demands. Most importantly, for prices below $\underline{\alpha}\bar{v}_i - \sigma^2x_{i0}^*$ the trader will buy the asset and for prices above $\bar{\alpha}\bar{v}_i - \sigma^2x_{i0}^*$ the trader will sell the asset.

We can now use these individual no-trade regions for prices to describe the conditions under which there will be an equilibrium with no-trade.¹¹ This will occur if the intersection of the trader's no-trade regions is non-empty. That is, if

$$\bigcap_{i=1}^I [\underline{\alpha}\bar{v}_i - \sigma^2 x_{i0}^*, \bar{\alpha}\bar{v}_i - \sigma^2 x_{i0}^*] \neq \emptyset. \quad (14)$$

This condition is described in Figure 1 in which the no-trade region of prices for trader i is denoted by $[\underline{p}^i, \bar{p}^i]$. The upper end point of this no-trade region of prices is determined by the trader who has the smallest price at which he would begin to sell the asset and the lower end point of the no-trade region is determined by the trader who has the highest price at which he would begin to buy the asset.

For the interesting case in which there is ambiguity about the shock to mean future values, $\underline{\alpha} < \bar{\alpha} < 1$, this condition for a no-trade equilibrium is equivalent to

$$(1 - \bar{\alpha})\text{Max}_i\{\bar{v}_i\} < (1 - \underline{\alpha})\text{Min}_i\{\bar{v}_i\}. \quad (15)$$

Theorem 1: Suppose that $\underline{\alpha} < \bar{\alpha} < 1$, then there is an equilibrium with no-trade if

$$\frac{\text{Max}_i\{\bar{v}_i\}}{\text{Min}_i\{\bar{v}_i\}} < \frac{1 - \underline{\alpha}}{1 - \bar{\alpha}}. \quad (16)$$

The left-hand-side of the no-trade inequality in Theorem 1 is the ratio of the most optimistic trader's mean to the least optimistic trader's mean. Thus, when there is more diversity in the population about the prior mean prices, an equilibrium with no-trade is more difficult to establish. In particular, when prior opinions are diverse, the portfolios that traders bring into period one are diverse, and thus there is more of an incentive for individual traders to move

¹¹ A similar approach is used in Bewley [2002] to characterize equilibrium with no-trade and in Shannon and Rigotti [2005] to characterize Pareto optimality and equilibrium.

toward the mean portfolio in response to a decline in expectations of future prices. The right-hand-side of the inequality (1.16) is the ratio of the largest possible percentage decline in mean prices to the smallest possible percentage decline in mean prices. This ratio measures the ambiguity about the percentage decline in future mean values. An increase in this ambiguity measure makes a no-trade equilibrium easier to establish.

Example 1: It may be useful to consider a simple example of this phenomenon. Suppose that there are three traders with prior means of $\bar{v}_{10} = 1, \bar{v}_{20} = 2,$ and $\bar{v}_{30} = 3$. Lets also suppose that in the most optimistic case the future value of the risky asset has not fallen; that is, $\bar{\alpha} = 1$. Then it follows from equation (1.16) that there will be no trade for any $\underline{\alpha} < 1$. Or to apply Theorem 1 directly, suppose that in the most optimistic case the mean future value has declined very little. Then, as long as in the least optimistic case the mean future value declines significantly there will be no trade. If, instead, there was no ambiguity about the decline in future value, there would be trade for any shock. For example, suppose that the traders agree that the mean future value of the risky asset has fallen by one-fourth. Then trader one would buy $(4\sigma^2)^{-1}$ shares of the risky asset, trader three would sell this number of shares and trader two would not trade.

2.3 Prices in the No-Trade Equilibrium

In this section we focus on an economy in which no trade occurs in response to the ambiguous shock to mean future asset values. That is, an economy in which the condition in Theorem 1 is satisfied. In this economy, there is no single price for the risky asset. Instead, the equilibrium is characterized by the following: (i) there is a range of prices at which no one is willing to trade the asset; and, (ii) for any price outside of this range, supply and demand for the

risky asset are not equal. So to understand valuation in this economy we need to look not at a single market clearing price, but at a range of prices at which no one, in fact, is transacting.

The end points of this range of prices are particularly interesting. The maximum price in this range is the lowest price at which some trader is willing to sell the risky asset. For any price greater than this price, someone wants to sell the asset, and no one wants to buy it. Thus, this price is a natural candidate for the *ask price* for the risky asset. If we were to ask each trader for the lowest price at which they would be willing to sell the risky asset and compute the minimum of these quotes, this ask price would be the result. Similarly, the minimum price in the no-trade range is the highest price at which some trader is willing to buy the asset. It is thus a natural candidate for the *bid price* for the risky asset. Again, if we were to ask each trader for the highest price at which they would be willing to buy the risky asset and compute the maximum of these quotes, this bid price would be the result.

Using the traders no-trade price intervals we see that the ask price is

$$\begin{aligned}
 ask &= \text{Min}_i[\bar{\alpha}\bar{v}_i - \sigma^2 x_{i0}^*] \\
 &= \text{Min}_i[\hat{v} - \sigma^2 \bar{x} - (1 - \bar{\alpha})\bar{v}_{i0}] \\
 &= \hat{v} - \sigma^2 \bar{x} - \text{Max}_i[(1 - \bar{\alpha})\bar{v}_{i0}]
 \end{aligned} \tag{17}$$

and the bid price is

$$\begin{aligned}
 bid &= \text{Max}_i[\underline{\alpha}\bar{v}_i - \sigma^2 x_{i0}^*] \\
 &= \text{Max}_i[\hat{v} - \sigma^2 \bar{x} - (1 - \underline{\alpha})\bar{v}_{i0}] \\
 &= \hat{v} - \sigma^2 \bar{x} - \text{Min}_i[(1 - \underline{\alpha})\bar{v}_{i0}].
 \end{aligned} \tag{18}$$

Thus, the bid price in this world is price set by the trader who is most optimistic about the largest possible decline in the value of the asset. Similarly, the ask price is set by the trader who is least optimistic about the smallest possible decline in the value of the asset. Clearly, if there is uncertainty, these prices are not the same, resulting in a bid-ask spread :

$$Spread = ask - bid = (1 - \underline{\alpha})Min_i\{\bar{v}_i\} - (1 - \bar{\alpha})Max_i\{\bar{v}_i\}. \quad (19)$$

It is straightforward to show that this spread is positive for an economy in a no-trade equilibrium.¹² This is an ambiguity spread in contrast to the asymmetric information spread usually studied in the market microstructure literature. With asymmetric information, the spread reflects the informational advantage that some traders have with respect to knowledge of the asset's true value. The asymmetric information spread arises because orders carry information and quotes to buy or sell are conditional expected values of the asset being traded given that an order to buy or sell arrives. A buy order signals good news, and thus the expected value of the asset conditional on a buy order, the ask price, is greater than the unconditional expected value of the asset. Similarly, a sell order signals bad news, and thus the expected value of the asset conditional on a sell order, the bid price, is less than the unconditional expected value of the asset. With asymmetric information the midpoint of the bid-ask spread is a reasonable approximation of the unconditional expected value of the asset.

In our model, no trader has an advantage over any other trader. There is no asymmetric information, and thus no learning from prices. Traders continue to disagree about the value of the asset (because of their heterogeneous prior beliefs), but more importantly they view the future value of the risky asset as being ambiguous. A spread arises because traders are unable to determine the valuation to attach to the asset given their uncertainty over the asset's possible values. What determines the size of the spread is the dispersion, or uncertainty, regarding the set of possible percentage declines in the expected value of the risky asset (i.e. α). If there were no uncertainty in this economy (that is, if $\underline{\alpha} = \bar{\alpha}$), then there would be no spread. Moreover, if

¹² Notice that this condition is essentially that of equation (1.16) which determines the existence of a no-trade equilibrium.

there were no uncertainty, then trade would actually occur, as risk alone, or even risk combined with asymmetric information, is not enough to preclude trading.¹³

Equilibrium in this uncertain economy is thus characterized by a range of prices, and trades actually occur at none of them. In effect, while nominal bid and ask prices can be calculated, the economy is in fact illiquid, with buyers and sellers unable, or perhaps better described as unwilling, to effectuate trades. Our model appears to capture many of the salient features of the current credit crisis, where uncertainty regarding the values to attach to complex financial assets such as mortgage derivatives has resulted in markets with quoted prices and no actual trading. We now turn to the question of how to assess the fair value of financial assets when uncertainty induces illiquidity.

3. Valuation in an Uncertain World

The issue of attaching “fair values” to assets has taken on particular importance due to accounting requirements imposed on firms by the Financial Accounting Standards Board. An excellent review of these requirements is given by Ryan [2008], but at a general level the rules require that firms determine “the price that would be received to sell an asset or transfer a liability in an orderly transaction between market participants at the measurement date”.¹⁴ The goal of such fair value accounting is to increase the transparency of asset valuation so that firms and their investors are better able to evaluate financial assets and make financial decisions.

While laudable as a goal, there are a variety of practical difficulties in implementing fair value

¹³ Of course, if asymmetric information is too large, then there may be no price at which a trade can occur and the market fails (see Akerloff [1970] or Glosten [1989]). However, in this case there would be no bid and ask prices and no functioning market of any kind. As demonstrated in microstructure models, the existence of asymmetric information more generally results in trades occurring at the bid and ask prices, and the spread reflecting an endogenous transactions cost.

¹⁴ See FAS 157 “Fair Value Measurements”.

accounting even under the best market conditions. Under the current conditions of the sub-prime crisis, Ryan argues that “this idea [of attaching a fair value] has become increasingly difficult to sustain even in thought experiments, and more importantly, practically useless as a guide to preparers’ estimation of fair values”.

Fundamental to the challenge of ascertaining fair values is how to impute values when markets may be illiquid or essentially incapable of providing transactions prices. FAS 157 specifies a hierarchy of approaches to establish fair value, but ultimately suggests using “the assumptions that market participants would use in pricing the asset or liability”¹⁵. It is in this context that our model can provide guidance as to how best to value assets in an uncertain world.

It is tempting to argue that the value of the risky asset is the highest price at which it could be sold. This is the bid price and so one line of thought is that this should be considered the fair value of the asset. The difficulty with this argument is that no one would actually be willing to sell the asset at this low price; instead, the lowest price at which anyone would sell the asset is the higher ask price. In “normal” market settings, the divergence between these two prices is small, and as trades are occurring at both prices, a bias in favor of the bid price does little harm. In our illiquid market with uncertainty, however, this presumption is not altogether clear.

One difficulty is that the bid price in our uncertain market does not have the standard properties that we typically find in market prices in more standard settings. The bid price here is set by the most optimistic trader about the worst possible outcome (essentially, it is one trader’s view of the “the best of a bad situation”). It is not an average valuation across traders, or even an average valuation across potential possible outcomes. Whether this price meets the FAS

¹⁵ FAS 157 classifies financial assets and liabilities into Level 1, Level 2, and Level 3 categories. Level 1 measurements use quoted market prices in active markets for identical assets for liabilities. Level 2 measurements use correlations with observable market based inputs, unobservable inputs that are corroborated by market data, or quoted market prices in markets that are not active. Level 3 measurements essentially rely on “marking to model” and use inputs that are significant and not corroborated with market data.

requirement of being an “exit” price is debatable since there is, in fact, no “orderly” (or for that matter, any) transaction occurring at this bid price. Conceptually, it is even more problematic to view this price as capturing fair value since it is focused only on the worst case outcome.

A related difficulty characterizes the ask price. The ask price is set by the least optimistic trader about the best possible outcome. Again, when uncertainty is large, there is no trade occurring at this price either. Basing fair market value on the most pessimistic individual’s valuation of the “worst of a good situation” does not seem consistent with the FAS requirement that valuations arise from “market participants who are knowledgeable, unrelated, and able to transact”.

How large a bias these difficulties impart to prices can be seen by comparing the bid or ask price with the notional price that we computed for scenario one in which only risk affected the asset value. Such a comparison is akin to asking what would be the price of an identical asset trading in an active market unaffected by uncertainty.¹⁶ To determine this, suppose that all traders were to agree that the mean future value of the risky asset has declined by some specific $\alpha \in [\underline{\alpha}, \bar{\alpha}]$. Then, from equation (1.5), the market price would be p_1^* and trade would occur at this price. This price represents the market’s consensus belief about the asset’s true value.

A natural conjecture is that the bid price in our model is “too low” because it is biased downward by the uncertainty. Comparison of this notional price with the bid price shows that this is generally, but not always, true. In particular, it is easy to show that $p_1^* \geq bid$ if and only if

$$(1 - \alpha)\hat{v} \leq (1 - \underline{\alpha})\text{Min}_i\{\bar{v}_{i0}\}. \quad (20)$$

¹⁶ Such a price would be consistent with a FAS 157 Level 1 designation (market prices in active markets for identical assets) as opposed to the Level 2 designation of the prices with uncertainty that reflect “quoted market prices in markets that are not active”.

So, provided prior beliefs are not too dispersed, the bid price is an underestimate of the notional, no-ambiguity, price as long as the actual decline in future value is not the worst possible case.

While this seems the natural case to consider, if these conditions do not hold, then it is possible for the bid price to be an overestimate of the notional price. This happens because, although the ambiguity results in a low bid price, price-setting shifts from the beliefs of the average trader (\hat{v}) to the beliefs of the most optimistic trader, imparting an upward bias to the price.

The other natural conjecture is that the ask price in our model is “too high” because it is biased upward by the uncertainty. Comparison of the notional price with the ask price shows that $ask \geq p_1^*$ if and only if

$$(1 - \alpha)\hat{v} \geq (1 - \bar{\alpha})Max_i\{\bar{v}_{i0}\}. \quad (21)$$

So, if there is little dispersion in prior beliefs then the ask price is an overestimate of the notional, no-ambiguity, price as long as the actual decline in future value is not the best possible case.

However, the ask price is not always biased upward from the notional price. Indeed, for the same reasons as discussed above, it is possible for the ask price to be an underestimate of the notional price. This can occur if the actual decline in future value is close enough to the best case and there is sufficient prior disagreement about the future value of the asset. In this case, although the ambiguity results in a high ask price, price-setting shifts from the beliefs of the average trader (\hat{v}) to the beliefs of the trader who is least optimistic about the smallest possible decline in the value of the asset.

If a market exists for an identical asset in which this notional price is available, then these biases in bid and ask prices suggest that fair value is better approximated by using the price from this alternative market. However, an unfortunate feature of the current sub-prime crises is that for some classes of assets, virtually all markets are characterized by the large spreads and lack of

trading characteristic of uncertainty. How then to best approximate this notional price using prices from these illiquid markets?

To address this question, note that the notional price will be between the bid and the ask prices if the inequalities we have analyzed above are both satisfied. That is, if

$$(1 - \underline{\alpha})\text{Min}_i\{\bar{v}_{i0}\} \geq (1 - \alpha)\hat{v} \geq (1 - \bar{\alpha})\text{Max}_i\{\bar{v}_{i0}\}. \quad (22)$$

In a no-trade equilibrium this interval exists as in this case the left-hand-side of the inequality above is greater than the right-hand-side. This will occur when the actual decline in the value of the risky asset lies between the best and worst cases and there is not too much dispersion in prior beliefs.

A natural candidate to use to value the asset, particularly if one believes the ask price is an overestimate and the bid price is an underestimate, is the midpoint of the bid ask spread. This approach has an obvious parallel with the use of the midpoint of the bid-ask spread in a world with asymmetric information as the best available estimate of the unconditional value of the asset. In microstructure models, the bid price is considered the appropriate exit price because it is the conditional expectation of the asset given that someone wishes to sell the asset. In our model, however, trades do not convey new information, and so the distinction between conditional and unconditional expected values is not meaningful. Consequently, the bid price does not play the role of a “best estimate” that it would in an asymmetric information environment.

In a world with uncertainty, this midpoint can, of course, be an under- or an over-estimate of the notional value of the asset. There is no distribution on the actual decline in the future value of the asset; if there were one, and traders agreed on it, then the asset would be priced according to the predicted distribution of future values. Nonetheless, it is tempting to ask

how the midpoint of the spread and the notional price compare if as an outside observer we impose some distribution on the decline in the value of the asset.

The Principle of Insufficient Reason (also known by the name John Maynard Keynes gave to it as the Principle of Indifference) may be useful in selecting a distribution to use here.¹⁷ This principle is an old idea which at its most basic level says that if an observer has no more information about each possible outcome than the name of the outcome then the observer should treat all outcomes as equally likely. This idea is most easily applied to a coin which appears to be fair in that heads and tails only differ in their names. In this case, it seems reasonable to assume that each has probability one-half. In our setting suppose that an outside observer knows only that the set of possible declines in the expected value of the risky asset is described by $1 - \alpha$ where $\alpha \in [\underline{\alpha}, \bar{\alpha}]$; that is, the observer knows only the set of names of the declines. Then the intuition behind the Principle of Insufficient Reason suggests that we use a uniform distribution on the set of possible declines, and so the mean decline would be $(\bar{\alpha} + \underline{\alpha}) / 2$.¹⁸

Using this idea, if one has to chose a number, it seems reasonable to suppose that on average the decline in the value of the asset is the simple average of the best and worst possible cases, i.e. $(\bar{\alpha} + \underline{\alpha}) / 2$. In this case, the relationship between the average notional price of the asset is greater than or less than the midpoint of the bid and ask as

$$(1 - \bar{\alpha})(\text{Max}_i \{\bar{v}_{i0}\} - \hat{v}) \stackrel{\geq}{\leq} (1 - \underline{\alpha})(\hat{v} - \text{Min}_i \{\bar{v}_{i0}\}). \quad (23)$$

¹⁷ Savage [1972], pages 63-67, provides a discussion of the intuition for this principle and the arguments against it.

¹⁸ Of course, this is equivalent to choosing a prior on the set of declines, and no prior is obviously superior to any other prior, or to representing the uncertainty with the entire set of distributions of mean future values. Our view is that a frequentist approach makes no sense here as this is not an experiment which has been, or really even could be, infinitely repeated. Instead, the uncertainty is necessarily subjective. This subjective uncertainty is perhaps not best represented by any distribution, but if one is forced to choose a prior then the uniform prior is at least as good as any other prior.

If there is no dispersion in prior beliefs, then the midpoint is equal to the average notional price of the asset.

This result suggests that in an uncertain world a better measure of fair value is given by the mid-point of the bid-ask spread. Note that our argument here is in terms of “better” and not “best”; because of the complex effects of uncertainty, every price (including the mid-point) is subject to some bias from “true values”.¹⁹ But the midpoint has the advantage that under reasonable conditions it is closest to the notional value, and hence seems more consistent with the motivations underlying the fair value concept.²⁰ Such an approach is similar to the standard view in microstructure models that the true value of the asset is approximated by the spread midpoint.

An alternative valuation approach is to eschew using market prices at all and simply rely on exogenous inputs to determine prices. In the current sub-prime crisis, a number of firms, most notably Citibank and UBS, relied on “marking to model” to value their CDO and mortgage derivative positions.²¹ A key input to this approach was the use of credit ratings to proxy for the quality (and therefore, value) of the specific tranches. Advocates of marking to model argue that when market prices are non-functional only exogenous data related to the underlying cash flows can provide reliable valuations.

The problem with this approach, of course, is that the exogenous inputs may not reflect the underlying shocks that have given rise to the valuation problems in the first place. Model

¹⁹ Indeed, if there is dispersion of prior beliefs, then the midpoint can be biased upward or downward from the average notional price of the asset. If average prior mean (\hat{v}) is the average of the maximum and minimum prior means, then calculation shows that the midpoint is an overestimate of the average notional price.

²⁰ That the midpoint may be the price most consistent with fair value is recognized by FAS 157 and by ASR (Accounting Series Release) 118, the guidance issued by the SEC on how funds should value securities in a portfolio. For example, as Ake and Hays [2007] note: “If the input used to measure fair value is based on bid and ask prices, the price within the bid-ask spread that is most representative of fair value in the circumstances should be used to measure fair value, regardless of where in the fair value hierarchy the input falls. FAS 157 does not preclude the use of mid-market pricing within a bid-ask spread.”

²¹ For more discussion, see “Why Citi Struggles to Tally Losses”, Wall Street Journal, Nov. 5, 2007, pg. C1.

prices based on stale data have proven to be a poor alternative to market-based valuations, a lesson illustrated by the extensive recent write-downs by most major “marking to model” banks. In uncertain markets, our model demonstrates while prices may be non-functional in the sense of reflecting transactions, they are not uninformative. This suggests that markets can, and should, play an important role in determining valuation even in an uncertain world.

4. Conclusions

We have investigated the implications of uncertainty for the liquidity and valuation of assets. In the context of a model in which traders have incomplete preferences over portfolios, we have demonstrated how an equilibrium can emerge in which bid and ask prices exist, a spread arises, but no trade actually occurs. In effect, markets “freeze” as uncertainty causes traders to revert to the status quo and trading ceases. In this illiquid world, quotes exist, but they do not have the properties we typically associate with bid and ask prices in normal, functioning markets. As we demonstrate, this has important implications for the on-going debate regarding how to determine fair values in illiquid markets.

While our model provides a cogent, and we believe useful, characterization of how uncertainty affects markets, it does require some structure to yield closed-form solutions. A natural issue is the extent to which our results would hold in a more general setting. One immediate extension is to allow our traders to differ on the set of possible declines (α) in the asset price. Because we allow our traders to have different prior means and we consider percentage declines in asset valuations, however, we believe this extension would have little effect on our results.

It would be interesting to enrich the analysis by allowing traders to anticipate the possibility of a decline in asset values and see how this affects their initial portfolios. Our traders would update as Bayesians (as we follow Bewley [2002] in relaxing completeness, rather than the independence axiom) so learning from shocks is straightforward. Similarly, it would be interesting to enrich the analysis to have real consequences of the choice of how to value the asset in period one. We discuss the ‘best way’ to do this, but so far in our model how it is done has no real consequences. Allowing either of these extensions might provide insights into the current controversy in accounting as to whether FASB-mandated write-downs are, in fact, the cause of subsequent asset price declines, rather than simply a representation of such effects (for research and discussion see Bloomfield, Nelson, and Smith [2006]; Ryan [2008]).

Finally, we have not discussed what equilibria would look like outside of the no-trade region. With incomplete preferences, equilibria need not be determinant. For each price outside of his no-trade region, each trader has a set of undominated demands and this can lead to a continuum of equilibria. Understanding these equilibria may lead to interesting insights about the evolution of market prices in uncertain environments.

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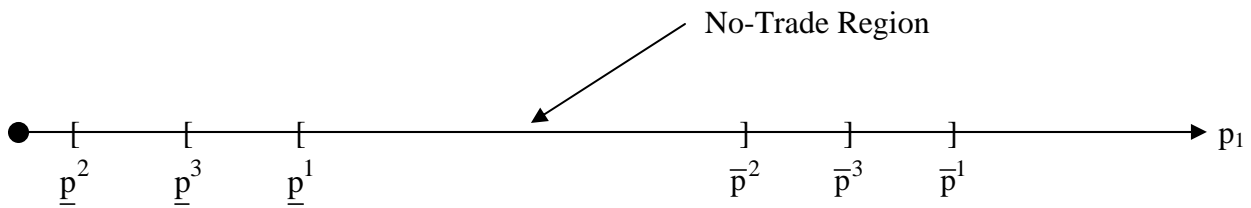
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Figure 1

The No-Trade Region of Prices



The interval of prices in which trader i will not trade is given by $[p^i, \bar{p}^i]$. The intersection of these intervals, the no-trade region, is $[p^1, \bar{p}^2]$.