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Probability and Statistics Applied to the Practice of Financial Risk Management: The Case of JP Morgan's RiskMetrics™

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PROBABILITY AND STATISTICS
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The Case of J P Morgan's RiskMetrics™

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This work describes applications of probability and statistics in RiskMetrics™, J P Morgan's methodology for quantifying market risk. The methodology implements an analytical approach to financial risk in trading, arbitrage, and investment based on the statistics of market moves in equities, bonds, currencies and commodities. The public unveiling of RiskMetrics™in October of 1994 attracted widespread interest among regulators, competing financial institutions, investment managers, and corporate treasurers, while the available technical documentation offers us a unique opportunity for informed statistical research on the theory and practice of financial risk management. For the purpose of identifying problems for further research, this discussion focuses on five applications of statistics in RiskMetrics™, which range from data analysis of daily returns and locally Gaussian processes to stochastic volatility models and Itô processes for the term structure of interest rates. The latter problems reflect the author's particular interest in stochastic inference for Markov processes and multivariate dependencies. Another important theme of this discussion, however, is devoted to attracting statisticians to the study of financial risk management and developing the foundations for collaborative work with financial economists and practicing risk managers. For this reason, this is also an expository document that touches several areas of active statistical research with applications to problems of risk management.

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1. INTRODUCTION

The decision in October of 1994 by J P Morgan to unveil RiskMetrics™—the risk-management product based on the bank's methodology for the management of financial risk—offered a unique opportunity to examine the development of the product and its impact on the market. RiskMetrics™methodology specifies an approach to quantifying market risks for the purpose of managing and controlling financial risk in trading, arbitrage and investment activities. This discussion describes the application of probability and statistics in RiskMetrics™with the purpose of identifying problems for further research, attracting statisticians to this line of investigation, and establishing a framework for collaboration with financial economists and managers of financial risk. While J P Morgan's view of financial risk is one among many, the advantage of this treatment lies with the opportunity to study specific statistical practices in RiskMetrics™and ultimately to focus our attention on open problems of broad interest to the field of risk management on which continuing research in probability and statistics can further progress.

RiskMetrics™methodology is concerned with the measurement of market risk in trading and investment, such as the exposure to changes in interest rates borne by investors in fixed-income instruments. If not properly managed, such exposure can have dire consequences, as testified to by recent events in Orange County, CA when the small but wealthy City of Irvine came calling for an expected $30 million return-on-investment in bonds for the purpose of paying its bills.

To asses risk for an investor with a position in a 30-year U.S. Government bond, for example, RiskMetrics™calculates the Value-at-Risk or VaR, a term that quantifies with 90% probability the maximum anticipated loss in value on the position due to an adverse movement in interest rates. For an investor with more diverse investments, consisting of positions in equities, bonds, commodities, and possibly foreign exchange, RiskMetrics™calculates the Value-at-Risk for each instrument separately and then aggregates their VaR's in a calculation of Daily-Earnings-at-Risk or DEaR, where portfolio effects or cross-instrument correlations play a critical role. Since the movements in the underlying instruments are modeled as random variables, VaR's and DEaR's are at heart statistical measures of market risk based on the properties of locally Gaussian processes. And although these measures of risk may be suitable for trading purposes with horizons as long as one day, they must be "rescaled" for the purpose of measuring investment risk, even for horizons as short as one week. To do so properly, scaling risk requires some statistical theory and an acute understanding of the processes underlying movements in financial markets.

Beginning with foundational questions of statistics in the problem of estimating means and variances and progressing to the general theory of Markov processes, there are many applications of probability and statistics in RiskMetrics™. The most basic problem is characterizing the properties of returns-on-investment and the inherent uncertainty of future returns. As financial markets evolve over time, however, the time series properties of returns on a portfolio of investments and the theory of locally Gaussian processes enter the equations of financial risk. The foremost problem remains to measure, manage, and control for risk due to adverse market moves. But since the variability of financial markets itself and so the rate of adverse moves appears to be a source of uncertainty, the so-called stochastic volatility models, including ARCH, GARCH and their descendants, have taken a central place in models of financial risk.

The general theory of processes directly enters the discussion of risk whenever positions in financial markets involve optionality and derivative securities. The theory of Itô processes, for example, underlies the valuation of derivative securities, as in the Black-Scholes formula, where the presence of optionality entails leverage, nonlinearities and liquidity issues that add significant layers of complexity to the assessment of market risk. The theory also underlies models for the term structure of interest rates, which plays a pervasive role in risk management because of the economic importance of interest rates. The term structure underlies assessments of risk in fixed-income, commodities, and foreign exchange. It also forms the basis of the cash-flow methodology for mapping financial instruments and calibrating their exposure to market risk.

These applications of the general theory of processes and locally Gaussian time series frame our interest in the statistics of risk management. In addition, RiskMetrics™Methodology anticipates or already employs applications of several other areas of active research in statistics, including hidden Markov models for stochastic volatility, the EM algorithm for data imputation, nuisance parameter problems for bonds with special features, the Bootstrap for scaling risk, and Monte Carlo methods for optionality, among others. Many of these applications are also active areas of econometric research in empirical financial economics, so the impact of such research for risk management reaches beyond RiskMetrics™and the origins of this case study.

This case study is an outgrowth of the field study by Phelan and Steele (1995) of the management of foreign exchange trading, conducted as part of The Wharton Financial Institutions Center Study of financial services for the Sloan Foundation. While the focus of the field study was the assessment of the state-of-the-art in the management of foreign exchange trading, we soon became aware of the importance of risk management systems, including statistical systems for quantifying market risk, in the successful operation of a foreign exchange trading desk. This identified an important area of risk management research having fundamental connections to applications of probability and statistics. J P Morgan's release of RiskMetrics™, which to our knowledge is the only publicly available document of its kind, was then a timely event in the evolution of this interest, making it possible to study the statistics of risk management as developed by practicing risk managers and the firm's market risk research group.

To organize our discussion, the next section describes the five applications of probability and statistics in RiskMetrics™considered here. The discussion is allotted specifically among problems in the statistical analysis of returns in the estimation of market risk, the time series properties and statistical description of volatility, a treatment of risk and optionality, and a methodology for mapping financial instruments that relies on an Itô process to model the term structure of interest rates. The discussion highlights relevant problems of estimation and inference from stochastic processes. Our primary source is J P Morgan's RiskMetrics™Technical Document, Third Edition, May 1995, a document of some 200 pages not included here, but we do include the firm's Introduction to RiskMetrics™, Third Edition, in Appendix A for informational purposes and access to their WEB site for RiskMetrics™Data and more documentation.

Section 3 draws specific observations and further outlines proposed directions for future research. These observations are necessarily selective, while the proposals lie principally in the direction of modeling and inference for stochastic processes. We conclude in Section 4 with some general remarks about this case study, the educational value of this work, and a view of where it fits in the broader framework for risk management research of Oldfield and Santomero (1994). Finally, the expository style of this document is intended to lay the groundwork for collaborative work between statisticians, financial economists and risk managers. To help disseminate terminology and share concepts, we compile a glossary of selected risk management terms in Appendix B and one of statistics in Appendix C.

2. STATISTICS IN RiskMetrics™

RiskMetrics™describes a methodology based on J P Morgan's approach to quantifying market risk in portfolios of fixed-income instruments, equities, foreign exchange, commodities, and their derivatives in the financial markets of 22 countries. It is an outgrowth of recent trends in financial markets—the growth of trading, securitization, derivatives, a focus on performance evaluation, indexing and the risk-return tradeoff in investing—and the product of over 15 years of developing a common framework for measuring market risk that is rooted historically in the pioneering work of Markowitz (1952) toward the modern portfolio theory. The practice of quantifying risk on the basis of value-at-risk developed along side of management practices over trading functions, where there is a need to mark-to-market trading positions frequently at prevailing prices and rates in order to project income over short horizons.

J P Morgan's RiskMetrics™consists of two main components: 1) the RiskMetrics™Data and 2) the RiskMetrics™Methodology. The data consists of spot prices or rates, volatilities, and correlation matrices for about 400 instruments in about 57 markets in 22 countries of Europe, North America, Japan, Australia and New Zealand. The data, which include both daily and monthly movements, are updated each trading day and are available over the internet at J P Morgan's Web site as well as from other vendors. For informational purposes and for additional documentation, we refer the reader to J P Morgan's Introduction to RiskMetrics™, Third Edition, available in Appendix A of this document.

This work treats the RiskMetrics™Methodology, which describes a statistical approach to the analysis, management, and control of market risk in trading, arbitrage and investment. From the framework for managing risk and the methods for quantifying risk, we describe five applications of probability and statistics applied to the practice of financial risk management in RiskMetrics™. While our discussion is drawn primarily from the RiskMetrics™Technical Document, Third Edition, May 1995, the following bibliographic references have also been helpful to our work. For a general introduction to risk management practices, we refer to the recent book on the subject by Smithson, Smith and Wifford (1994), while we refer to Hull (1993) for a treatment of the theory and practice of financial engineering and aspects of fixed-income securities, essential parts of investment and risk management. Finally, for theoretical treatments of continuous-time finance and dynamic asset pricing theory and derivative securities, we refer the reader to the advanced texts of Merton (1990) and Duffie (1992), two works requiring a background in the general theory of Markov processes at the level of Karatzas and Shreve (1988).

I. Returns and Risk

We introduce the statistical description of (compound) returns on financial investment and the role of the Gaussian distribution and variance to characterize risk. An adverse market move then emerges as 1.65 times an estimated standard deviation, reflecting that 1.65 is the 95% quantile of the standard Normal distribution, and ultimately defining the Value-at-Risk or VaR on an investment position.

To fix ideas on a two period model, let S_0 denote the initial spot price of a commodity, yield on a government bond, or exchange rate between two currencies. Let S_1 denote the spot price one unit of time later, where the sampling interval is typically one day. As a measure of the movement in the underlying instrument, the compound return R is given by

$$(1) \qquad R = \log(S_1) - \log(S_0) = \log(1 + \triangle^{\%} S),$$

where $\triangle^{\%} S$ denotes the relative change in spot price, namely $(S_1 - S_0)/S_0$ over the sampling interval. In the statistics of risk management, and there is some empirical evidence for this assumption, we may suppose that R is a Gaussian random variable having location μ_0 and scale σ_0, so that

$$(2) \qquad I\!P(R \leq z) = \Phi(\mu_0, \sigma_0;\, z) = \frac{1}{\sigma_0\sqrt{2\pi}} \int_{-\infty}^{z} \exp -(r - \mu_0)^2/2\sigma_0 dr,$$

for every z in $I\!R$. The expected return is the mean μ_0 and, in the context of a zero-sum investment, the Sharpe ratio is μ_0/σ_0.

For a trader or investor, a realization of R then essentially models a percentage change or market move in spot price, yield or rate of currency exchange. To define an adverse market move, we have the well-known equation

$$(3) \qquad I\!P(|R - \mu_0| > 1.65\sigma_0) = 0.10,$$

since 1.65 is the 95% quantile of the standard Normal distribution. The quantity $1.65\sigma_0$ defines an adverse market move. It is the maximum percentage change, essentially, to be expected of the underlying instrument with 90% probability. As movements to the tails of the distribution of compound returns, adverse market moves are associated with extreme values and events of small probability.

An adverse market move in trading is an inauspicious change in the value of a trader's position, calibrated here by the quantity $1.65\sigma_0$. For a trader with a long position in an instrument that initially trades at price S_0, an adverse market move is one to the left-hand tail of the distribution of R, namely to the point $\mu_0 - 1.65\sigma_0$ or lower. All things equal, therefore, adverse market moves occur at rate 5% or one in twenty trading days. Otherwise, if S_0 refers to the initial yield on a fixed-income instrument with modified duration D_0, the corresponding price risk for a long position in the instrument is given by $1.65\sigma_0 S_0 D_0$, which scales the yield risk for the sensitivity or derivative of the price of the instrument with respect to interest rates, while the risk still lies in inauspicious moves of size $1.65\sigma_0 S_0 D_0$ or more to the left of expectation.

For short positions, naturally, the risk lies in adverse market moves to the right-hand tail of the distribution of compound returns. Nevertheless, we calibrate short-position risk with the same statistic because of the symmetry of the Gaussian distribution and, implicitly, the loss function.

Value-at-Risk or VaR simply scales the adverse market move for the size of the trading or investment position. This calculation in the RiskMetrics™methodology thereby explicitly assumes that the underlying μ_0 is zero, which is argued as reasonable for short horizons such as a day. For a U.S. Dollar investor in a hypothetical 15-year zero coupon bond with initial yield 7.35%, market value US \$50 million and modified duration 9.11, the Value-at-Risk in the position over one trading day is given by

$$\text{VaR} = \text{US } \$(50)1.65\sigma_0(0.0735)(9.11), \tag{4}$$

where $1.65\sigma_0(7.35)(9.11)$ calibrates the maximum percentage price change expected with 90% probability. For traders long or short this position, they can expect to lose VaR U.S. Dollars or more with probability 5% in one trading day. Otherwise, for a spot position of

US \$250 million in German Marks (DEM), the Value-at-Risk over one trading day is given simply by

$$\text{VaR} = \text{US } \$250(1.65\sigma_0), \tag{5}$$

where $100(1.65\sigma_0)$ calibrates the maximum percentage change in the DEM/USD exchange rate expected with 90% probability. The VaR for a US \$250 million equity position, say, is also given by Equation 5, upon substituting for the sigma belonging to the equity.

Anticipating our next concern, recall that a forward contract on DEM is priced at the forward rate, a rate of foreign exchange depending on the initial spot rate of exchange and the domestic or U.S. interest rate and the foreign or German interest rate. To calculate the VaR on a US \$250 million forward position then requires the price VaR of Equation 5 and two interest-rate VaR's, one domestic and one foreign, of the kind in Equation 4. The three VaR's are then aggregated into the Daily-Earnings-at-Risk, a multivariate calculation to which we turn now.

To wit, consider the initial price P_0 of some commodity or equity, the initial yield Y_0 on a 30-year U.S. Government bond, and the initial spot exchange rate between the German Mark (DEM) and the U.S. Dollar (USD). After one trading day, these will have price P_1, yield Y_1, and spot exchange X_1, say, and substituting P, Y, and X for S in Equation 1, their respective compound returns are denoted R^p, R^y, and R^x. We suppose that these returns have expectations μ_0^p, μ_0^y, and μ_0^x and standard deviations σ_0^p, σ_0^y, and σ_0^x, respectively.

Now, let R and μ_0 denote the vectors in $\mathbb{R}^3$ given by

$$R = (R^p, R^y, R^x) \qquad \text{and} \qquad \mu_0 = (\mu_0^p, \mu_0^y, \mu_0^x), \tag{6}$$

representing market moves and their expectations for the price, yield and spot exchange rate. In the statistics of risk management, where there is again some empirical evidence for this assumption, we may suppose that the vector R is a Gaussian random variable in $\mathbb{R}^3$ having location μ_0 and scale $\Sigma_0^{\frac{1}{2}}$. The covariance matrix Σ_0 is given by

$$\Sigma_0 = \Lambda_0 \mathcal{P}_0 \Lambda_0, \tag{7}$$

where Λ_0 denotes the diagonal matrix $\text{diag}(\sigma_0^p, \sigma_0^y, \sigma_0^x)$ and $\mathcal{P}_0$ denotes the correlation matrix belonging to R. For ρ_0^{xy} the correlation between market moves in the spot exchange rate and the yield, the covariance is given by $\rho_0^{xy}\sigma_0^x\sigma_0^y$, so the correlation calibrates the effect of simultaneous positions in the bond and the currency. It is also the basis of the calculation of Daily-Earnings-at-Risk.

For example, for $(\rho_0^{xy})^2 \neq 1$, the distribution of the pair (R^x, R^y) has density ϕ given by the equation

$$\phi(\mu_0^x, \mu_0^y, \sigma_0^x, \sigma_0^y, \rho_0^{xy};\ w, z) = \frac{1}{2\pi\sigma_0^x\sigma_0^y\sqrt{1-(\rho_0^{xy})^2}} \exp -\frac{1}{2} Q(\mu_0^x, \mu_0^y, \sigma_0^x, \sigma_0^y, \rho_0^{xy};\ w, z), \tag{8}$$

where the quadratic form Q is given by

$$Q(\mu_0^x, \mu_0^y, \sigma_0^x, \sigma_0^y, \rho_0^{xy};\ w, z) = \frac{1}{1-(\rho_0^{xy})^2}\Big[\frac{(z-\mu_0^x)^2}{(\sigma_0^x)^2} + \frac{(w-\mu_0^y)^2}{(\sigma_0^y)^2} - 2\rho_0^{xy}\Big(\frac{z-\mu_0^x}{\sigma_0^x}\Big)\Big(\frac{w-\mu_0^y}{\sigma_0^y}\Big)\Big],$$

for every z and w in $I\!\!R$, the law of a Gaussian variate in the plane with degree of linear association ρ_0^{xy}. We therefore have the stochastic representation

$$\begin{aligned} R^y &= \mu_0^y + \sigma_0^y W^{(1)} \\ R^x &= \mu_0^x + \rho_0^{xy}\frac{\sigma_0^x}{\sigma_0^y}(R^y - \mu_0^y) + \sigma_0^x\sqrt{1-(\rho_0^{xy})^2}\,W^{(2)}, \end{aligned} \tag{9}$$

where the innovations $W^{(1)}$ and $W^{(2)}$ are independent Gaussian random variables with mean zero and scale 1. Notice that the representation for R^x is the classical regression of the movement in the exchange rate on the movement of the yield, expressing the critical role of the correlation in managing risk by way of diversifying portfolios or hedging offsetting risks. If the squared correlation is 1, however, then R^x is almost surely proportional to R^y, meaning lockstep movements of the exchange rate and the yield. Thus perfectly correlated, the instruments carry statistically indistinguishable risk.

Next, the return R^b on the price of the underlying bond has the representation

$$R^b = -D_0 Y_0 R^y, \tag{10}$$

where D_0 is initially the modified duration of the bond and Y_0 the initial yield. Therefore, the pair (R^b, R^x) has the stochastic representation

$$\begin{aligned} R^b &= \mu_0^b + \sigma_0^b W^{(1)} \\ R^x &= \mu_0^x + \tilde{\rho}_0^{xb}\frac{\sigma_0^x}{\sigma_0^b}(R^b - \mu_0^b) + \sigma_0^x\sqrt{1-(\tilde{\rho}_0^{xb})^2}\,W^{(2)}, \end{aligned} \tag{11}$$

where the coefficients μ_0^b, σ_0^b, and $\tilde{\rho}_0^{xb}$ satisfy the equation

$$\mu_0^b = -D_0 Y_0 \mu_0^y \qquad \sigma_0^b = D_0 Y_0 \sigma_0^y \qquad \text{and} \qquad \tilde{\rho}_0^{xb} = -\rho_0^{xy}.$$

The change in sign for the correlation reflects the inverse relationship between yields and prices of bonds. This representation, in terms of movements in the prices of instruments, is the basis for calibrating market risk from adverse price moves in the underlying bond and currency.

Daily-Earnings-at-Risk is a value-at-risk for a portfolio position. For this calculation the RiskMetrics™methodology again explicitly assumes the underlying μ_0's are zero. For a trader with positions in a bond and a currency, let VaR^b and VaR^x denote the respective Value-at-Risk as determined by Equations 4 and 5. To distinguish long from short positions, let δ^b equal +1 whenever the trader is long the bond and δ^b equal -1, otherwise. For the currency, let δ^x be so defined by the trader's position in foreign exchange. The Daily-Earnings-at-Risk or DEaR is then given by

$$\text{DEaR} = \sqrt{(\text{VaR}^x)^2 + (\text{VaR}^b)^2 + 2\delta^x\delta^b\tilde{\rho}_0^{xb}\text{VaR}^x\text{VaR}^b}, \tag{12}$$

which is the Value-at-Risk belonging to the two-instrument portfolio consisting of a δ^x currency position and a δ^b bond position, long or short as the case may be.

Notice that if the trader's positions and the market correlations satisfy the equation

$$\tilde{\rho}_0^{xb} = \delta^x\delta^b,$$

namely the configuration of positions and correlations with highest risk, then the implications of Equation 11 and a little algebra at Equation 12 give the expression

$$\text{DEaR} = \text{VaR}^x + \text{VaR}^b. \tag{13}$$

This gives the so-called additive case; an unfortunate choice of language that statisticians would have reserved for the case $\tilde{\rho}_0^{xb} = 0$. The additive case produces a conservative measure of risk, which denies the benefits of diversification, that is presently recommended by the Basel Committee of The Bank of International Settlements for setting capital requirements. This recommendation may change in future.

In general, the initial market positions and the adverse market moves determine a Value-at-Risk for each instrument, say VaR^p, VaR^b, and VaR^x for the price risk in the commodity, the bond, and the currency. These are determined according to the formulae of Equations 4 and 5, the former converting yield risk to price risk in dollar terms. To reflect the change from yield risk in the representation at Equation 9 to price risk in the representation at Equation 11, we introduce the matrix $\tilde{\mathcal{P}}_0$ for the correlation matrix belonging to the triple (R^p, R^b, R^x), specifying a price-risk correlation matrix which adjusts the signs on the correlations in the original correlation matrix $\mathcal{P}_0$ for the inverse relationship between movements in yield and movements in price for bonds. The value-at-risk vector V is given by

$$V = (\mathrm{VaR}^p, \mathrm{VaR}^x, \mathrm{Var}^b)^\top, \tag{14}$$

the $\top$ denotes transpose. To distinguish long from short positions in each instrument, we introduce δ^p, δ^b, and δ^x and the diagonal matrix Δ given by

$$\Delta = \mathrm{diag}(\delta^p, \delta^b, \delta^x), \tag{15}$$

used below to rotate the correlation matrix with respect to the trader's positions as at Equation 12. The Daily-Earnings-at-Risk is then given by the equation

$$\mathrm{DEaR} = \sqrt{V^\top \Delta \tilde{\mathcal{P}}_0 \Delta V}, \tag{16}$$

which is the projected U.S. Dollar loss from adverse market moves confronting the trader's position one trading day hence. For correlations perfectly at odds with the trader's position, set

$$\tilde{\rho}_0^{px} = \delta^p \delta^x, \quad \tilde{\rho}_0^{pb} = \delta^p \delta^b \quad \text{and} \quad \tilde{\rho}_0^{xb} = \delta^x \delta^b$$

for the correlations in the matrix $\tilde{\mathcal{P}}_0$ in Equation 16. This configuration of correlations and positions produces the highest risk, defining again the additive case for which DEaR is given by the sum of the VaR's.

As an exercise, consider a currency trader with a forward USD position in the market for DEM. The forward rate F_0 is given by

$$F_0 = X_0 \frac{1 + r_0^f}{1 + r_0^d},$$

where X_0 is the initial spot rate of exchange, r_0^f the German interest rate, r_0^d the U.S. interest rate. The movements in forward rates are linear in movements in spot exchange rates and, essentially, the two interest rates. Identify the parameters required to calculate the trader's VaR in the forward position by way of Equation 16. □

II. Market Moves and Locally Gaussian Processes

We introduce the statistical analysis of multivariate time series, such as the 400 time series monitored in the RiskMetrics™Data set, in the context of locally Gaussian processes. This introduces the notions of local characteristics of market moves, the infinitesimal description of risk, and the span of financial markets.

For each $t \geq 0$, let t denote today's date and let S_t denotes the spot prices, rates and yields on a set of financial instruments. For components of S_t, we have $S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(d)}$, where the dimension d is the number of instruments. Naturally, S_t belongs to a history $\mathcal{H}_t$ recording all historical events available to the market participants at time t. Though the metaphor of market information does nicely here, these histories generate a filtration $\mathbf{H} = (\mathcal{H}_t), t \geq 0$ of market information in the technical sense of an increasing family of sub-sigma fields belonging to some probability space.

The RiskMetrics™Data set has dimension 400, representing equity indices, interest rates at standardize vertices along the yield curve, exchange rates in U.S. Dollar terms, and futures-based commodity prices drawn from markets here and there. Section D of The RiskMetrics™Technical Document, Third Edition, May 1995 gives a complete description of the market coverage. Naturally, the choice of time series to monitor is a critical one in deriving a picture of market risk, as is the quality of the estimate of the yield curve.

For the purpose of modeling risk, the time series $t \to R_t$ of compound returns takes values in $\mathbb{R}^d$ with components given by

$$(17) \qquad R_{t+1}^{(j)} = \log(S_{t+1}^{(j)}) - \log(S_t^{(j)}) = \log(1 + (\triangle^{\%} S^{(j)})_{t+1}), \qquad j = 1, \ldots, d,$$

where $(\triangle^{\%} S^{(j)})_{t+1}$ denotes the relative change in spot price, namely $(S_{t+1}^{(j)} - S_t^{(j)})/S_t^{(j)}$ over the sampling interval $(t, t+1]$. The random variable R_{t+1} then models the market movements for d instruments one day hence from date t.

Next, let W denote a Wiener process on $\mathbb{R}^m$, where the components of W represent m orthogonal sources of uncertainty underlying the market. For each $t \geq 0$, suppose that A_t takes values in the space of lower triangular $d \times m$ matrices. In the statistics of risk management, we may suppose that the return process $t \to R_t$ satisfies the equation

$$R_{t+1} = \mu_t + A_t(W_{t+1} - W_t), \tag{18}$$

which is the natural analog of the representation at Equation 9. Since increments of W in unit time are Gaussian with mean 0 and identity covariance, the return process generates a locally Gaussian spot process $t \to \log S_t$ having local characteristics $t \to \mu_t$ and $t \to \Sigma_t$ given by the conditional expectations

$$\mathbb{E}_{\mathcal{H}_t} R_{t+1} = \mu_t \qquad \text{and} \qquad \mathbb{E}_{\mathcal{H}_t}(R_{t+1} - \mu_t)(R_{t+1} - \mu_t)^\top = A_t A_t^\top = \Sigma_t, \tag{19}$$

specifying the infinitesimal mean and the infinitesimal covariance given the history $\mathcal{H}_t$ at the close of trading day t.

Notice that this specifies the spot process $t \to S_t$ in terms of a random infinitesimal generator of a stochastic flow on $\mathbb{R}^d$, see Kunita (1990). Notwithstanding, we recognize this as the usual theoretical framework of equilibrium pricing models and optimal consumption and portfolio allocation, see Duffie (1992) or for a probabilist's view see Karatzas (1990) . As the span of the market is plausibly greater than that of the d monitored instruments, we have necessarily an incomplete picture of market risk. In this sense, the number and mix of instruments in a risk management system is critical to its ability to reach as many of the sources of uncertainty generating the returns process of Equation 17.

Adverse market moves, Value-at-Risk, and Daily-Earnings-at-Risk are determined each trading day by the local characteristics, while the local characteristics are drawn from the history of market moves. However this estimation is achieved, and referring to Equation 7, we again write

$$\Sigma_t = \Lambda_t \mathcal{P}_t \Lambda_t, \tag{20}$$

where Λ_t is a diagonal matrix $\text{diag}(\sigma_t^{(1)}, \ldots, \sigma_t^{(d)})$ of infinitesimal standard deviations and $\mathcal{P}_t$ is the infinitesimal correlation matrix belonging to R_{t+1}. At the close of trading day t, the diagonal of the matrix $1.65\Lambda_t$ calibrates the scale of adverse market moves for the complement of d instruments.

Notice, however, that some of these instruments refer to yields, so for the purpose of calibrating price risk for cash positions in bonds, it still remains to scale their adverse market moves for current modified duration and yield. We refer, for example, to Equation 4. Of course, this has implications for sign changes in the underlying correlation matrix for the purpose of calibrating price risk in portfolio positions.

For each $j = 1, 2, \ldots, d$ and $t \geq 0$, let $\text{VaR}_t^{(j)}$ denote the Value-at-Risk for instrument j a the close of trading day t. The trader is long or short the instrument as $\delta_t^{(j)}$ is +1 or -1, respectively. Assemble the $\text{VaR}_t^{(j)}$'s in the vector V_t and the $\delta_t^{(j)}$'s as elements of the diagonal matrix Δ_t. The underlying price-risk correlation matrix is denoted $\tilde{\mathcal{P}}_t$ having elements $\tilde{\rho}_t^{(jk)}$, $1 \leq j, k \leq d$. In this case, the projected Daily-Earnings-at-Risk or DEaR_t is given by the equation

$$(21) \quad \text{DEaR}_t = \sqrt{V_t^\top \Delta_t \tilde{\mathcal{P}}_t \Delta V_t} = \sqrt{\sum_{j=1}^{d} (\text{VaR}_t^{(j)})^2 + 2 \sum_{j=1}^{d-1} \sum_{k=j+1}^{d} \delta_t^{(j)} \delta_t^{(k)} \tilde{\rho}_t^{(jk)} \text{VaR}_t^{(j)} \text{VaR}_t^{(k)}},$$

representing the U.S. Dollars at risk of loss from adverse market moves over the sampling interval $(t, t+1]$, one trading day hence. For conservative risk management and, for now, for regulatory purposes, the additive case again sets $\tilde{\rho}_t^{(jk)} = \delta_t^{(j)} \delta_t^{(k)}$ in Equation 21, so that all instruments are correlated perfectly at odds with the trader's position. The additive-case DEaR_t is given by the sum

$$(22) \qquad \text{DEaR}_t = \sum_{j=1}^{d} \text{VaR}_t^{(j)},$$

the sum of the individual position VaR's as recommended by The Basel Committee.

The question of estimation remains. So does the question of scaling these daily estimates of risk to longer investment horizons, say 6 to 8 weeks. We treat this subject next in the context of stochastic volatility models. □

III. Stochastic Volatility

Volatility or movement in financial markets exhibits features of a random process of a persistent nature punctuated by episodes of accelerated, correlated movements to the tails of empirical distributions. From this widespread observation emerged one of the more important innovations in modeling financial time series, the introduction of stochastic volatility models. Stochastic volatility models, such as ARCH and GARCH, can improve forecasting, but they also entail significant challenges in filtering and estimation. While such models may provide rationale for the exponential smoothers used to estimate volatility for the purpose of risk management in The RiskMetrics™Data set, this area remains an active one of research.

To fix ideas, consider a basket of three currencies: the Japanese Yen (JYN) and the German Mark (DEM) relative to the U.S. Dollar (USD). For each trading day t, let $t \to X_t^y$ denote the JYN/USD spot exchange rate, $t \to X_t^m$ the DEM/USD spot exchange rate. The spot processes have return processes $t \to R_t^y$ and $t \to R_t^m$, respectively determined by Equation 17. Referring to Equation 18 and Equation 9, we have the representation

$$
\begin{aligned}
R_{t+1}^y &= \mu_t^y + \sigma_t^y (W_{t+1}^{(1)} - W_t^{(1)}) \\
R_{t+1}^m &= \mu_t^m + \rho_t^{my} \frac{\sigma_t^m}{\sigma_t^y} (R_t^y - \mu_t^y) + \sigma_t^m \sqrt{1 - (\rho_t^{my})^2} (W_{t+1}^{(2)} - W_t^{(2)}),
\end{aligned}
\tag{23}
$$

where the innovations $W^{(1)}$ and $W^{(2)}$ are independent Wiener processes, $t \to \sigma_t^m$ and $t \to \mu_t^m$ the local characteristics of the DEM/USD exchange rate, $t \to \sigma_t^y$ and $t \to \mu_t^y$ the local characteristics of the JYN/USD exchange rate, and $t \to \rho_t^{my}$ the infinitesimal correlation between the two.

The RiskMetrics™Data set monitors such movements in exchange rates and provides measures of volatility based on estimates of the local characteristics of the returns process drawn from the history of exchange rate movements. Since this involves markets on two continents and one distant island, there are timing issues which are addressed in the details of Section D of The RiskMetrics™Technical Document. In addition, pieces of data may be missing for one or another reason, so the missing data are imputed by an application of the EM algorithm.

The seminal work on time series models for volatility belongs to Engle (1982), who introduced the class of autoregressive conditional heteroskesdastic or ARCH models, although the literature as surveyed by Bollerslev, Chou, and Kroner (1992) is now voluminous. A popular variation on ARCH, the so-called generalized autoregressive heteroskedastic or GARCH models, were introduced by Bollerslev (1986) and Taylor (1986). To illustrate ideas here simply, we treat only the apparently most empirically successful model GARCH (1,1), which contains

1 ARCH component and 1 component of the generalization. As another simplification, we treat only the so called diagonal form of the model for the three currency problem described above.

We suppose that the second local characteristics $t \to (\sigma_t^m, \sigma_t^y, \rho_t^{my})$ are random processes adapted to the history $(\mathcal{H}_t)$, $t \geq 0$, an implicit assumption all along. For the innovations of the returns process of Equation 23, we introduce the processes $t \to \eta_t^m$ and $t \to \eta_t^y$ given by the equations

$$\eta_{t+1}^m = R_{t+1}^m - \mu_t^m \quad \text{and} \quad \eta_{t+1}^y = R_{t+1}^y - \mu_t^y, \tag{24}$$

the deviations of currency movements from conditional expectation given $\mathcal{H}_t$. For a diagonal GARCH (1,1) model, we suppose that the second local characteristics satisfy the equations

$$\begin{aligned} (\sigma_t^m)^2 &= a^m + b^m(\eta_{t-1}^m)^2 + c^m(\sigma_{t-1}^m)^2 \\ (\sigma_t^y)^2 &= a^y + b^y(\eta_{t-1}^y)^2 + c^y(\sigma_{t-1}^y)^2 \\ \rho_t^{my} &= a^{my} + b^{my}\eta_{t-1}^m\eta_{t-1}^y + c^{my}\rho_{t-1}^{my}, \end{aligned} \tag{25}$$

where the a's, b's and c's are fixed parameters satisfying classical constraints, so that these equations specify well-defined, wide sense stationary second local characteristics.

Through the incorporation of additional lags, feedback between equations, and transformations to other scales, there are many variations on the theme suggested by Equation 25. One such variation sets the b's and c's equal to 1 and subtracts terms proportional to lagged innovations belonging to time $t-2$, giving a particular diagonal GARCH (2,1) model and an ARIMA (0,1,1) structure to the second characteristics themselves. As may be verified, therefore, the second local characteristics are forecast optimally by single exponential smoothing.

The RiskMetrics™Data set gives estimates of volatility based on estimates of the second local characteristics derived by single exponential smoothing of the squared innovations of the returns. For sampling intervals at one trading day or shorter, we suppose the first local characteristic, namely μ^m and μ^y, is zero; this is routinely done for all of the time series in the RiskMetrics™Data set. The innovations η^m and η^y are then equal to the returns R^m and R^y. For $1 > \lambda > 0$ and for sufficiently large K, let $t \to (\hat{\sigma}_t^m, \hat{\sigma}_t^y, \hat{\rho}_t^{my})$ denote the empirical processes given by

$$
(26) \qquad
\begin{aligned}
\hat{\sigma}_t^m &= \sqrt{\lambda(\hat{\sigma}_{t-K}^m)^2 + (1-\lambda)\sum_{s=t-K+1}^{t} \lambda^{t-s}(R_s^m)^2} \\
\hat{\sigma}_t^y &= \sqrt{\lambda(\hat{\sigma}_{t-K}^y)^2 + (1-\lambda)\sum_{s=t-K+1}^{t} \lambda^{t-s}(R_s^y)^2} \\
\hat{\rho}_t^{my} &= \lambda\hat{\rho}_{t-K}^{my} + \frac{(1-\lambda)}{\hat{\sigma}_{t+1}^m\hat{\sigma}_{t+1}^y}\sum_{s=t-K+1}^{t} \lambda^{t-s}R_s^m R_s^y,
\end{aligned}
$$

which is a one-step ahead forecast based on single exponential smoothing of the squared innovations over a stretch of days in the interval $[t-K, t]$, where the initial values $\hat{\sigma}_{t-K}^m$, $\hat{\sigma}_{t-K}^y$, and $\hat{\rho}_{t-K}^{my}$ are given by the ordinary sample standard deviations and correlation over the same stretch of time. The parameter K is chosen so that $(1-\lambda)\sum_{l\geq K}\lambda^l$ is sufficiently small, say less than 1% for $\lambda = 0.94$ and $K = 74$ days, or about one quarter.

For each day t, the triple $(\hat{\sigma}_t^m, \hat{\sigma}_t^y, \hat{\rho}_t^{my})$ forecasts the triple $(\sigma_t^m, \sigma_t^y, \rho_t^{my})$ for the purpose of forecasting movements over the sampling interval $(t, t+1]$, one trading day hence. The anticipated benefit of exponential smoothing relative to simple averages, of course, is that such filters are reasonably responsive to the most current profile of movements in the market, responding rapidly to punctuated movements and decaying quickly as they fade from view.

For the purpose of risk management, The RiskMetrics™Data set reports daily volatilities $t \to V_t^m$ and $t \to V_t^y$ given by

$$
(27) \qquad V_t^m = 1.65\hat{\sigma}_t^m \quad \text{and} \quad V_t^y = 1.65\hat{\sigma}_t^y,
$$

which are empirical adverse market moves as defined in Section I: Return and Risk. It also reports the empirical daily correlations $t \to \hat{\rho}_t^{my}$. The idea is for the user to construct daily estimates of VaR's and DEaR's.

In the parlance of risk management, the volatilities of Equation 27 are called ex-ante volatilities. There are also ex-post volatilities, defined in RiskMetrics™as ordinary sample standard deviations taken over historical stretches of trading days. Ex-post volatilities come into play when scaling ex-ante volatilities from daily forecasts to forecasts of monthly volatility, taken to mean 25 trading days, used for the purpose of risk management over longer investment horizons.

Scaling is an important issue in applications of statistics to risk management. One proposal in RiskMetrics™is to apply the root-time rule to a series of ex-post daily volatilities, followed by an exponential filter to estimate ex-ante monthly volatility. But institutional features of some markets and shifting volatilities spell problems for the root-time rule, so an alternative proposal in RiskMetrics™explores an application of Efron's Bootstrap. In this application, the user subsamples daily innovations for the purpose of inducing a Bootstrap empirical distribution of monthly ex-post volatilities. While not perfect, it avoids some questionable assumptions and affords the user a look at the probable sampling distribution of monthly volatility. □

IV. Risk and Optionality

Optionality, such as a callable bond or a put on oil futures contract, adds significantly to the complexity of assessing market risk. The value of an option, say as computed by the Black-Scholes formula, is affected nonlinearly by changes in volatility of the underlying instrument and the short rate of interest, not to mention the added risks associated with leverage and liquidity. In some cases, when the option is at or near the money and the underlying security behaves locally like Brownian motion, approximations based on the "greeks" may perform reasonably well. Otherwise, full evaluation may be necessary where some form of Monte Carlo technique comes into play.

The most pervasive and accessible option pricing formula is Black-Scholes. For $t \geq 0$, a trader with U.S. Dollars buys the option, but not the obligation, to buy German Marks (DEM) for the negotiated strike price x, say, at some specified future time $T > t$. This is a call option. We suppose that the spot exchange rate is S_t and the "volatility" is σ_t^m. The risk-free rate of interest for the problem is r_t. In this case, the price V^c or value of the call option is given by the formula

$$V^c(r_t, \sigma_t^m, x, S_t, t, T) = S_t \Phi(d) - x\Phi(d - \sigma_t^m \sqrt{T-t}), \tag{28}$$

where Φ is the standard Normal distribution function, $T - t$ the time to expiration of the option, and d the parameter given by

$$d = \frac{\log(S_t) - \log(x) + (r_t + (\sigma_t^m)^2/2)(T-t)}{\sigma_t^m \sqrt{T-t}}.$$

On the other hand, the price V^p of a put, as the option but not the obligation to sell DEM at the strike price x at time T, is given by the formula

$$V^p(r_t, \sigma_t^m, x, S_t, t, T) = x\mathrm{e}^{-r_t(T-t)}\Phi(\sigma_t^m \sqrt{T-t} - d) - S_t\Phi(-d), \tag{29}$$

which may be derived by an appeal to put-call parity. Before turning to issues in risk management for positions in options, we state briefly the origins of this formula.

The Black-Scholes formula arises from an application of Itô's formula for transformations of Wiener processes and Girsanov's formula for locally absolutely continuous change of measure, see Duffie (1992) or Karatzas and Shreve (1988). Under the risk-neutral measure, which equates the underlying appreciation rate of the currency to the risk free rate r_0, the spot price $t \to S_t$ is given by the Geometric Brownian motion

$$S_t = S_0 \exp\, \sigma_0^m W_t + (r_0 - (\sigma_0^m)^2/2)t, \tag{30}$$

where W is a Wiener process, so that the discounted exchange rate $t \to \tilde{S}_t$,

$$\tilde{S}_t = S_t \exp -r_0 t,$$

is a martingale. The payoff of the call option is the positive part of $S_T - x$, so that the value of the option of Equation 28 is then given by the conditional expectation

$$V^c(r_0, \sigma_0^m, x, S_0, 0, T) = \hat{\mathbb{E}}_{S_0}(S_T - x)^+ \exp -r_0 T, \tag{31}$$

the conditional expected payoff given S_0 under the risk neutral measure. Notice that Equations 28 and 29 are applications of this equation, upon setting S_0 to S_t, r_0 to r_t and σ_0^m to σ_t^m and assuming these rates prevail for a Geometric Brownian motion over (t, ∞). Notice that this story may also apply to any non-dividend-paying stock, while the story for securities with gains is more complex.

The option then represents a derivative security among the trader's positions in the financial markets. Unlike our earlier exercise with currency forwards, however, the movements of the value of the option are nonlinear functions of the market moves, making it more challenging to calibrate market risk satisfactorily; for some derivative securities, liquidity issues and highly leveraged positions only complicate the task. For the purpose of risk management, therefore, RiskMetrics™estimates the Value-at-Risk in an option position by parametric approximation based on the so-called greeks or by full valuation based on a structured Monte Carlo technique. The greeks refer to a selected set of coefficients from the Taylor series expansion of the option price. The key greeks are δ, Γ, Θ, Λ (called vega), and ρ for sensitivity or the derivative of V^c with respect to the initial exchange rate, the initial exchange rate again, the time to expiration of the option, volatility, and the interest rate, respectively. These are used to approximate volatility in the option price due to volatility in the inputs.

For example, the price of the option responds to market moves over the sampling interval $(t, t+1]$, meaning changes in the spot exchange rate, the volatility of the currency, and the short rate of interest. At the close of trading day t, for example, we suppose that the underlying exchange rate, not the risk-neutral construct above, satisfies the equation

$$R_{t+1} = \log(1 + (\Delta^{\%} S)_{t+1}) = \sigma_t^m (W_{t+1} - W_t), \tag{32}$$

the underlying locally Gaussian approximation to the returns process over $(t, t+1]$, one trading day hence; one can evaluate this approximation explicitly for the exchange rate process described above. We then suppose that the price change of the call option on the currency satisfies the approximation

$$V^c(r_t, \sigma_t^m, x, S_{t+1}, t+1, T) - V^c(r_t, \sigma_t^m, x, S_t, t, T) = \delta_t S_t R_{t+1} + \frac{1}{2} S_t^2 \Gamma_t (R_{t+1})^2, \tag{33}$$

the Delta-plus-Gamma Equivalent for the move in option price from movement in the underlying exchange rate, only, where

$$\delta_t = \frac{\partial}{\partial y} V^c(r_t, \sigma_t^m, x, y, t, T)_{|y=S_t} \quad \text{and} \quad \Gamma_t = \frac{\partial^2}{\partial y \partial y} V^c(r_t, \sigma_t^m, x, y, t, T)_{|y=S_t}.$$

The standard deviation σ_t^v of the right-hand side of Equation 33 satisfies the equation

$$\sigma_t^v = \sqrt{S_t^2 \delta_t^2 (\sigma_t^m)^2 + \frac{1}{2} (S_t \Gamma_t (\sigma_t^m)^2)^2}, \tag{34}$$

since the cross-product term vanishes by virtue of the local Gaussian approximation at Equation 32. In this case, the quantity $1.65\sigma_t^v$ calibrates the Delta-plus-Gamma Equivalent

Adverse Market Move in the price of the option from movements in the exchange rate. If the trader has a US \$150 million position at the close of trading day t, then

$$\text{VaR} = \text{US } \$150(1.65)\sigma_t^v,$$

the Delta-plus-Gamma Equivalent Value-at-Risk on the position over $(t, t+1]$, one trading day hence.

As an exercise, implement these calculations for the Black-Scholes formula. Then add the greek term ρ for interest rate sensitivity to the approximation at Equation 33. The approximated change in the option price now depends on the local characteristics belonging to the pair of processes $t \rightarrow (S_t, r_t)$, see for example the representation at Equation 23. Show that the Value-at-Risk on the option from market moves in the exchange rate and the interest rate emerges from a calculation of Daily-Earnings-at-Risk.

For options at or near the money, parametric approximations to the Black-Scholes formula work reasonably well. In other cases, RiskMetrics™applies a full valuation approach based on a structured Monte Carlo technique. Here are the steps.

Step 1. Determine a RiskMetrics™volatility matrix Σ based on d instruments,

Step 2. Provided Σ is strictly positive definite, let G denote the Cholesky triangle got by Cholesky factorization of Σ, so G is lower triangular with positive diagonal entries and $\Sigma = GG^\top$,

Step 3. Generate a Gaussian random variable X in $I\!R^d$ having mean zero and identity covariance matrix,

Step 4. Compute $Y = GX$, which generates an innovation return and a Gaussian random variable in $I\!R^d$ having covariance Σ,

Step 5. Set $\tilde{S} = F \exp Y$, componentwise, where F is a vector in $I\!R^d$ representing expected future spot rates and prices, the vector $\exp Y$ representing a simulated price relative,

Step 6. Evaluate the value of the option with inputs $\tilde{S}$ and calculate the change from its initial value, cf. the left-hand side of Equation 33 with $S_{t+1} = \tilde{S}$, Step 7. Specify a number N and iterate Steps 1 through 6 as many times, thereby generating a Monte Carlo sampling distribution of option price changes,

Step 8. Estimate the 5% quantile from the Monte Carlo sample of option price changes obtained in Step 7.

For trading day t, say, the covariance matrix Σ in Step 1 refers to Σ_t of Equation 19 or more usually the RiskMetrics™estimate as at Equation 26. The Y at Step 4 simulates an approximate return R_{t+1}, one day hence, for the d instruments in the sense of Equation 18, but with the μ's set to zero as is the practice in RiskMetrics™. With reference to Equations 17 and 18, the spot prices satisfy

$$S_{t+1} = S_t \exp R_{t+1} = S_t(1 + (\Delta^{\%} S)_{t+1}),$$

componentwise, so that $\tilde{S}$ simulates the spot price, one day hence. Finally, at the end of this ride, the 5% quantile from Step 8 is the estimated adverse market move for the price of the option and one proceeds with it to calculate the Value-at-Risk. □

V. Mapping and Term Structure of Interest Rates

In June 1995 an investor owns US $100,000 of U.S. Government Treasuries yielding 6.75% and maturing in July 2005. The underlying note pays coupons of US $6,750 over the next ten years plus the principal. One of these cash flows is maturing in 6.25 years, which falls between two standard vertices at 5 and 7 years along the yield curve. Since volatilities are available only at standard vertices, the 6.25 year cash flow will have to be mapped in some way to the standard vertices in order to calibrate risk.

The RiskMetrics™Data set monitors some 400 or so time series covering equity indices, yields at standard vertices, money rates, swap rates, foreign exchange, and commodity futures prices available in the markets of North America, Europe, Australia, New Zealand and Japan. This standard set of instruments attempts to span as much of the uncertainty or volatility in financial markets as is possible, but it can not reach it all nor cover every instrument at hand. Mapping describes a methodology for representing any instrument in terms of a standardized set of instruments, monitored daily, in a risk management system.

First, we discuss a simple example in a familiar context that illustrates one of the tradeoffs entailed by mapping risk in equity positions to the span of the market indices. Consider an equity position with market value MV^{e} in an equity having price volatility σ^{e}. The Value-at-Risk or VaR in the position is then given by

$$\mathrm{VaR} = \mathrm{MV}^{\mathrm{e}}(1.65)\sigma^{\mathrm{e}}, \tag{35}$$

where the quantity $(1.65)\sigma^{\mathrm{e}}$ calibrates the adverse market move. By virtue of the Capital Asset Pricing Model (CAPM), the one period return on the equity R^{e} is given by the regression equation

$$R^{\mathrm{e}} = a + \beta R^{\mathrm{m}} + \eta, \tag{36}$$

where R^{m} is the return on the market, β and a are equity specific regression coefficients, and η the equity specific innovation having mean zero and scale σ'. The volatility for the equity, therefore, is given by the equation

$$\sigma^{\mathrm{e}} = \sqrt{\beta^2(\sigma^{\mathrm{m}})^2 + (\sigma')^2}, \tag{37}$$

where σ^{m} is the market volatility. Since market volatility is monitored by RiskMetrics™'s equity index series and not that of individual equities, the Value-at-Risk on the equity position is taken to be the quantity VaR′ given by the equation

$$\mathrm{VaR}' = \mathrm{MV}^{\mathrm{e}}\beta(1.65)\sigma^{\mathrm{m}}, \tag{38}$$

a mapping that perhaps mistakes the risk to the extent that the equity specific risk σ' is not diversified away.

The balance of this treatment of mapping concentrates on instruments bearing primarily interest rate risk, such as positions in government bonds, interest rate swaps, and forward rate agreements. While there are various methods based on duration, principal and cash flow for mapping fixed-income instruments, cash flow maps is the method of choice in the risk management framework of RiskMetrics™. We treat examples of this practice here, concentrating ultimately on the critical dependence of the methodology on J P Morgan's Term Structure Model and theory of the term structure of interest rates. This theory underlies the estimate of the yield curve used to value risk of cash flow equivalents on bonds, and other interest-rate sensitive securities, while the estimation of the yield curve relies statistically on an application of the general theory of Markov processes and a penalized least-squares problem involving nuisance parameters.

Mapping Cash Flows for Bonds

RiskMetrics™monitors volatilities for yields on zero coupon bonds at standardized set of vertices, namely 2 years, 3 years, 4 years, 5 years, 7 years, 9 years, 10 years, 15 years, 20 years, and 30 years, along the yield curve; shorter rates are monitored by money market rates. For the hypothetical US $100,000 U.S. Treasury Note position described at the open of this section, the cash flow of US $6,750 maturing in 6.25 years must be mapped to the standard set of zero rates for the purpose of calibrating risk.

In analogy with the so-called barbell trade, the present value of the 6.25 year cash flow is allocated proportionally to the 5 and 7 year vertices. For purpose of calibrating risk, the weights are chosen so as to preserve the total risk in the underlying cash position, thus preserving the VaR belonging to the position. Once allocated to standard vertices, the cash flow equivalents can then be used in later calculations of DEaR's involving positions in other instruments as well.

For example, let Z_5 and Z_7 denote the zero rates at the 5 and 7 year vertices, respectively. Interpolating linearly along the yield curve between the points $(5, Z_5)$ and $(7, Z_7)$, the zero rate $Z_{6.25}$ for 6.25 years is given by the equation

$$Z_{6.25} = qZ_5 + pZ_7, \qquad 0 < p < 1, \;\; q = 1 - p, \tag{39}$$

where specifically $p = .625 = (6.25 - 5)/(7 - 5)$. Given the 5 and 7 year yield volatilities σ_5^y and σ_7^y, the 6.25 year yield volatility $\sigma_{6.25}^y$ is given by the equation

$$\sigma_{6.25}^y = \sqrt{q^2(\sigma_5^y)^2 + p^2(\sigma_7^y)^2 + 2qp\rho_{5,7}\sigma_5^y\sigma_7^y}, \tag{40}$$

where $\rho_{5,7}$ is the underlying correlation between movements in the 5 and 7 year yields[1]. As at Equation 4, the underlying price volatility $\sigma_{6.25}^p$ for the interpolated zero is given by the

[1]RiskMetrics™simply interpolates here, taking $q\sigma_5^y + p\sigma_7^y$ for $\sigma_{6.25}^y$, seemingly at variance with the representation at Equation 39.

equation

$$\sigma^p_{6.25} = D_{6.25} Z_{6.25} \sigma^y_{6.25}, \tag{41}$$

where $D_{6.25}$ is the modified duration of the interpolated zero, namely $D_{6.25} = 6.25/(1+Z_{6.25})$. Although this quantity would appear to be suitable for the purpose of calibrating risk in a cash position maturing at year 6.25, The RiskMetrics™Methodology proceeds to allocate the cash flow across positions in the bonds maturing 5 and 7 years hence. This splits the cash flow between the two standard vertices, whose volatilities and correlations are monitored daily, for later use in calculations of DEaR's involving a portfolio of instruments. The proportions are chosen to preserve the price volatility $\sigma^p_{6.25}$ as determined by Equations 39, 40, and 41. Preserving price volatility, of course, preserves the underlying position's VaR.

To wit, the cash flow US \$ 6,750 has present value $\mathrm{MV}_{6.25}$ given by the equation

$$\mathrm{MV}_{6.25} = \mathrm{US}\ \$6.750(1 + Z_{6.25})^{-6.25}. \tag{42}$$

The problem is to allocate this value across the 5 and 7 year vertices. That is, for $1 > \alpha > 0$ and $\beta = 1 - \alpha$, consider the allocation $\alpha\mathrm{MV}_{6.25}$ to the 5 year vertex and $\beta\mathrm{MV}_{6.25}$ to the 7 year vertex. In so doing, the cash Value-at-Risk on the 6.25 year cash flow on the bond position is then given by the equation

$$\mathrm{VaR}(\alpha) = \mathrm{MV}_{6.25} 1.65 \sigma^p_{6.25}(\alpha), \tag{43}$$

where the scale $\sigma^p_{6.25}(\alpha)$ is the price-risk volatility given by the equation

$$\sigma^p_{6.25}(\alpha) = \sqrt{\alpha^2(\sigma^p_5)^2 + \beta^2(\sigma_7)^2 + 2\alpha\beta\rho_{5,7}\sigma^p_5\sigma^p_7}, \tag{44}$$

where $\rho_{5,7}$ is the correlation between the movements in yield at the 5 and 7 year vertices. Notice that the price volatilities σ^p_5 and σ^p_7 for the 5 and 7 year bond prices appear in Equation 44, given analogously to Equation 41 by the underlying yield volatilities σ^y_5 and σ^y_7, the corresponding modified durations D_5 and D_7, and the yields Z_5 and Z_7. Notice also that the price correlation $\rho_{5,7}$ in Equation 44 is the same as the yield correlation in Equation 40.

In contrast to allocation based on duration mapping, the allocation ratio α is chosen here to preserve the underlying price volatility on the cash position as determined by Equation 41.

That is, equate $\sigma^p_{6.25}$ of Equation 41 to $\sigma^p_{6.25}(\alpha)$ of Equation 44 and solve for α in the resulting quadratic equation:

$$a\alpha^2 + b\alpha + c = 0, \tag{45}$$

where the coefficients a, b and c are given by

$$\begin{aligned} a &= (\sigma_5)^2 + (\sigma_7)^2 + 2\rho_{5,7}\sigma_5\sigma_7 \\ b &= 2\rho_{5,7}\sigma_5\sigma_7 - 2\sigma_7 \\ c &= (\sigma_7)^2 - (\sigma^p_{6.25})^2. \end{aligned}$$

When a solution exists, take the root inside the unit interval that preserves the sign of the cash flows. Usually, if the price volatility curve is upward sloping, take the root α^*,

$$\alpha^* = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

otherwise take the other root; but we urge the reader to digest the table of contingency rules on page 113 of The RiskMetrics™Technical Document, Third Edition, May 1995.

As an example, take the one in The RiskMetrics™Technical Document, Third Edition, May 1995 on page 115. The position is similar to the one described above, but for FFr100,000 in a 7.5% French Government bond. First, interpolate a FFr7,500 cash flow maturing at 6.08 years between $Z_5 = 7.628\%$ and $Z_7 = 7.794\%$ to get $Z_{6.08} = 7.717\%$. The corresponding yield volatilities are $\sigma^y_5 = 0.01503$, $\sigma^y_7 = 0.01374$, and $\sigma^y_{6.08} = 0.01434$ by the RiskMetrics™practice of interpolation. The correlation $\rho_{5,7} = 0.962$. Multiplying these yield volatilities by the corresponding yield and the modified duration (maturity over 1 plus yield) of the zero to get the price volatilities of $\sigma^p_5 = 0.00533$, $\sigma^p_7 = 0.00696$, and $\sigma^p_{6.08} = 0.00624$ for the underlying cash equivalents. The price volatility curve is upward sloping, so take the root $\alpha^* = 0.4073$. The market value of the original cash flow FFr7,500 is FFr4,774, so allocation gives the attendant cash flows of FFr1,950 to the 5 year vertex and FFr2,824 to the 7 year vertex. Determine the Value-at-Risk on the position. For a U.S. Dollar investor, a full treatment would include foreign exchange risk. As an exercise, formulate this problem within The RiskMetrics™Framework. Do so for the hedged versus the unhedged foreign exchange exposure. Show that the allocated cash flow equivalents are needed to calculate the Daily-Earnings-at-Risk.

As a second example, suppose a trader sells a 6×12 month forward rate agreement on US \$1 million at 7.24%. This locks in the investment rate for six months, starting of course six months from now. Since the rate 7.24% is determined by the 6 month and 12 month money market rates, the money market rate vertices in The RiskMetrics™Data set are used to map and calibrate the risk. The position is equivalent to borrowing US \$1 million at a discount for six months and investing the proceeds for 12 months. This equivalence determines how the position is treated for the purpose of valuing risk.

At inception, treat the agreement as a trading position that is short one instrument, given by the 6 month money rate vertex, and long one instrument, given by the 12 month money rate vertex. The position has one negative 6 month cash flow of US \$1 million and one positive 12 month cash flow determined by principal plus return on 12 month investment of the US \$1 million principle as discounted by the annualized 6 month rate. The Value-at-Risk for the forward rate agreement is then given by the Daily-Earnings-at-Risk at Equation 12, using the money rate 6 and 12 month price volatilities and the corresponding correlation.

One month hence, however, the 6 month arm of the trade falls between the 3 month and the 6 month money rate vertices, while the 12 month arm falls between the 6 and 12 month money rate vertices. The two arms of the trade are therefore mapped separately, as described above, one between the 3 and 6 month vertices and the other between the 6 and 12 month vertices. The updated Value-at-Risk is then Daily-Earnings-at-Risk for a three instrument trade involving the price volatilities at the 3, 6, and 12 month vertices of the money market rates.

In each example, then, the three step algorithm is to first characterize the underlying cash flows belonging to an instrument or position, second to identify the equivalent cash flows in the basis of monitored instruments, and third to map the underlying cash flows to their equivalents for the purpose of quantifying the Value-at-Risk. On the one hand, the methodology depends critically on an algorithm equating price risk in the underlying cash position. On the other, it depends critically on the term structure of interest rates. We now turn to the problem of estimation of the yield curve, where, from a statistical point of view, there emerges an application of Itô processes and a sophisticated least-squares problem.

The bond example above referred to zero rates belonging to hypothetical zero coupon Government bonds maturing at times given by the standard vertices. These yields are not directly observed, but inferred from the prices on liquid bonds trading in the respective Government bond market. They are essential to the workings of the cash flow methodology for mapping cash flows of bonds.

RiskMetrics™attempts to span interest-rate risk in 22 markets by monitoring volatilities and correlations of money market rates, swap rates and yields on zero coupon bonds. The volatilities for money market rates are intended to cover all short term interest-rate risk, including Treasury bills, the prime rate and commercial paper rates. The volatility of swap yields are intended to cover interest-rate swaps, loans and non-governmental bonds, and credit-sensitive securities because they include some spread for credit risk, although all of these are treated as AAA quality. Given N swap-rate vertices and an initial zero swap rate Z_1, the remaining zero swap rates $Z_2, Z_3, \ldots, Z_N$ are derived from the published swap rates $S_1, S_2, \ldots, S_N$, recursively using the formula

$$1 = \sum_{k=1}^{m-1} \frac{S_k}{(1+Z_k)^k} + \frac{1+S_m}{(1+Z_m)^m}, \qquad m = 2, \ldots, N, \tag{46}$$

a no arbitrage condition on the present value of unit investment at prevailing swap yields. The inferred zeroes, their volatilities and correlations are used to calibrate risk for credit sensitive securities according to the algorithm described above.

The volatilities of government zeroes are synthesized statistically from observable bond prices, bond valuations, and a model for the term structure of interest rates. The term-structure model originates with the work of Cox, Ingersoll and Ross (1985), who devise a one-factor model featuring an Itô process for the risk-neutral short rate of interest. All of these ingredients, finally, go into a least-squares problem for the purpose of fitting the underlying synthetic zero coupon rates. Although this discussion is taken from The RiskMetrics™Technical Document, Third Edition, May 1995, a more complete discussion of J P Morgan's Term Structure Model is found in Murphy (1992) and Fody (1992).

Anticipating the details of the fitting procedure, we notice that the least-squares problem is constrained so that the forward rates implied by the fitted zeroes adhere to a mean reverting forward rate curve. This constraint may be viewed classically as a smoothness penalty or as arising from a posterior distribution of the zeroes given the observed bond prices and the term structure of interest rates. Another aspect of the problem is the presence of nuisance parameters arising from special features of the bonds.

The yield to maturity or internal rate of return is a measure of the interest rate on a bond. It is defined as the rate of discount on remaining coupons and principle that equates the present value of a bond to its price. Because coupon-bearing issues have payments before maturity, their internal rates of return fail to provide a suitable benchmark for pricing non-governmental issues or for calibrating volatility in the term structure of interest rates. For this purpose, RiskMetrics™employs a statistical procedure that infers the yield curve.

For today's date $t > 0$ and a later date $T > t$, let $Y_t(T - t)$ denote the yield on a hypothetical zero-coupon bond maturity in time $T - t$ hence. The price $B_t(T - t)$ of the bond is given by the equation

$$B_t(T - t) = (1 + Y_t(T - t))^{-(T-t)}, \tag{47}$$

so that $Y_t(T - t)$ defines the rate of growth of a $\$B_t(T - t)$ investment at date t that is redeemed for \$1 at date T. The collection of such yields over a standard set of vertices, for example $Y_t(k)$ for $k = 2$ years, 3 years, $\ldots$, 20 years, 30 years, determines the term structure of interest rates for date t. The zero rates would be used to calibrate market risk for interest-rate sensitive securities, as Z_5 and Z_7 were used above. Since zero rates are not observed directly, the problem then is to estimate them from data on coupon-bearing bonds.

For a bond with no special features, coupons of \$C on each of the dates T_1, T_2,. . ., T_m, and principle \$P redeemable on date T_m, the theoretical price Q_t of the bond in given by the equation

$$Q_t = \sum_{k=1}^{m} \mathrm{C} B_t(T_k - t) + \mathrm{P} B_t(T_m - t), \tag{48}$$

the present value of its future cash flows given the internal rates of return on the zero coupon bond. The market price P_t of the bond, as determined by the bond market, is used to fit the underlying zero rates by minimizing the distance between the market price and the present value of the bond.

Many bonds actually come with special features or with liquidity premia attached to them. For example, the yields on French Government OATs reflect their relative liquidity to BTANs. The internal rates of return on such bonds need to be adjusted for these features before they can be combined in an estimate of the zero rates. As we see below, the theoretical price of bonds with special features includes bond-specific components that act as a kind of nuisance parameter in the least-squares problem for fitting zero rates. To exhibit this valuation, however, we first need the so-called forward rates or expected future short rates of interest.

The forward rates are derived from the term structure of interest rates of Equation 47 and an arbitrage condition expressed by the principle of equal expected return on the buy-and-hold strategy versus the roll-over strategy for two period investments horizons. For today's date $t > 0$ and two later dates $T > S \geq t$, the forward rate $f_t(S-t, T-t)$ is given by the equation

$$f_t(S-t, T-t) = \frac{(S-t)\ln(1+Y_t(S-t)) - (T-t)\ln(1+Y_t(T-t))}{T-S}, \tag{49}$$

expressed for convenience in the form of continuous compounding rates. The term structure then implies the rate of interest $f_t(S-t, T-t)$ on investment in a zero coupon bond issued at date S for a period $T-S$ hence. These rates are used below to price bonds where $S-t$ and $T-t$ fall on adjacent, standard maturity vertices along the yield curve.

The behavior of the forward rates as derived from the fitted zero rates is an important consideration in evaluating the quality of the fit. Conventional arguments have it that forward rates exhibit mean reversion, reflecting an economically cyclic demand for credit on the part of borrowers. Expectations of such behavior are also part of conventional critiques of models for the term structure of interest rates. For the purpose of fitting zero rates here, mean reverting forward rates and specifically those implied by the term structure model of Cox, Ingersoll and Ross (1985) constrain the set of admissible solutions to the least-squares problem below.

The valuation of a sample of n bonds proceeds as follows. Suppose that today's date is $t > 0$ and that there are N vertices to the term structure. For each k, $k = 1, \ldots, N$, let $v(k)$ denote the maturity of the vertex k; the first vertex might be 2 years, and so on. For the ith bond, $i = 1, \ldots, n$, let $C^{(i)}(k)$ denote the cash flow of the bond accruing at date $t + v(k)$.

To account for special features, such as sinking-fund provisions, let M denote the number of yield spreads. For each l, $l = 1, \ldots, M$, let $s_t(l)$ denote the yield spread and $b^{(i)}(l)$ the ith bond's exposure to the lth spread. The valuation or theoretical price $Q_t^{(i)}$ of bond i is then given by the equation

$$(50)\; Q_t^{(i)} = \sum_{k=1}^{N} C^{(i)}(k) \exp -[\sum_{j=1}^{k} f_t(v(j-1), v(j))(v(j) - v(j-1)) + v(k) \sum_{l=1}^{M} b^{(i)}(l) s_t(l)],$$

where $v(0) = 0$ and the f_t are the forward rates implied by the term structure at Equation 49. The first term in the exponent of every summand determines the discount on any cash flow accruing at dates forward along the yield curve. The second term in the exponent indicates a special feature of the bond. For bonds with sinking-fund provisions, the $b^{(i)}(l)$ indicate (that is, by taking value 1 or 0) exposure to the market spread $s_t(l)$ for sinking funds[2]. For the purpose of fitting zero rates, special features are nuisance parameters arising from heterogeneity among bonds in the market.

For $i = 1, \ldots, n$, let $P_t^{(i)}$ denote the market price of the ith bond. The principle of least squares will give an estimate of the term structure zero rates by minimizing the distance between the $P_t^{(i)}$ and the $Q_t^{(i)}$ of Equation 50. But this naive fit tends toward highly variable forward rates, not sufficiently smooth given the conventional arguments for mean reversion. An alternative approach is to add mean reversion as a constraint on the original least problem. RiskMetrics™does this essentially, but the constraint is given in terms of a penalty for goodness of fit to the forward rates implied by the Cox, Ingersoll and Ross (1985) model for the term structure of interest rates.

In particular, let W denote a Wiener process and a, b, and σ positive constants. In the Cox-Ingersoll-Ross (CIR) model for the term structure, the risk-neutral, short rate of interest R is given by an Itô process satisfying the Itô stochastic differential equation

$$(51) \qquad \mathrm{d}R_t = a(b - R_t)\mathrm{d}t + \sigma\sqrt{R_t} W(\mathrm{d}t),$$

which yields as solution a positive interest-rate process that is mean reverting to level b at rate a. This process is used to set prices on bonds from which the internal rates of return

[2]Yield spreads are premia offered by firms when borrowing funds from investors with the promise of proportioned redemption over the life of the bond.

determine the term structure and so the forward rates of interest, see for example Chapter 15 of Hull (1993), particularly Section 15.12 and Equation 15.24.

The forward rates implied by the CIR model are the prior forward rates that constrain the least-squares problem in RiskMetrics™. Given an anchor vertex $v(k')$, for some $0 \leq k' < N$, and the forward rate $f_t(v(k'), v(k'+1))$, the prior forward rates ζ_t implied by the CIR model are given by the equation

$$(52) \quad \zeta_t(v(k), v(k+1)) = \frac{\alpha(k)}{\alpha(k')}[f_t(v(k'), v(k'+1)) - \beta(k')] + \beta(k), \qquad k = 0, \ldots, N-1,$$

where the coefficients α and β are given by the equations

$$\alpha(k) = \frac{1}{v(k+1) - v(k)}[\frac{\exp v(k+1)\theta}{\theta + \vartheta(\exp[v(k+1)\theta] - 1)} - \frac{\exp v(k+1)\theta}{\theta + \vartheta(\exp[v(k)\theta] - 1)}]$$

$$\beta(k) = \frac{2ab}{(\sigma)^2}[\frac{1}{v(k+1) - v(k)}\log[\frac{\theta + \vartheta(\exp[v(k+1)\theta] - 1)}{\theta + \vartheta(\exp[v(k)\theta] - 1)}] - \vartheta],$$

with the parameters θ and ϑ given by

$$\theta = \sqrt{(a+\varrho)^2 + 2(\sigma)^2} \qquad \text{and} \qquad \vartheta = \frac{\theta + a + \varrho}{2}.$$

The parameter ϱ denotes a market risk factor, which enters the CIR coefficients in the RiskMetrics™treatment. These coefficients are determined by interpolating the CIR forward rates of Equation 15.24 in Hull (1993) through the point $(v(k'), f_t(v(k'), v(k'+1)))$, using linearity of the CIR forward rates with respect to the short rate of interest[3]. The parameters $a + \varrho$ and σ are estimated from the historical volatility of interest rates by equating historical volatilities of changes in spot interest rates (i.e. zero coupon rates) with their derived volatilities under the CIR model at two maturity vertices. For this, compare C.2.21 of The RiskMetrics™Technical Document with Equations 15.23 and 15.25 in Hull (1993).

Finally, the forward rate prior, bond valuations and market prices enter the objective function G as given by the equation

$$(53) \quad G(f_t, s_t) = \sum_{i=1}^{n} \left(\frac{P_t^{(i)} - Q_t^{(i)}}{Q_t^{(i)}} \right)^2 + \phi \frac{n}{N} \sum_{k=1}^{N} [f_t(v(k-1), v(k)) - \zeta_t(v(k-1), v(k))]^2,$$

[3]This treatment differs from that in RiskMetrics™, where two anchors are used in an interpolation with obscure derivation.

where the $Q_t^{(i)}$ are given by Equation 50, the ζ_t by Equation 52, the $P_t^{(i)}$ by market prices. Minimizing G over the forward rates f_t and the yield spreads s_t gives the least squares estimate of the term structure of interest rates by way of Equation 49. The external parameter ϕ determines the penalty for adherence of the forward rates to the prior forward rates, though one may apparently argue this problem from a Bayesian perspective of estimating a posterior mode[4]. Empirically, RiskMetrics™uses ϕ of 1/2 for estimating zero rates in the government bond market of France.

In summary, the elements of the methodology consist of a sample of market prices for bonds, a model for the term structure of interest rates, and an objective function for fitting the synthetic zero rates. The term structure model is used for bond valuation and to fixed the forward rate prior. The prior forward rates, or some natural smoothness criterion, enter the objective function to penalize departures of the fitted term structure from conventional or modeled expectations about the behavior of forward rates. In the end, the fitted zeroes, their volatilities and correlations are used to map and value the risk of interest-rate-sensitive securities. We gave examples above. □

This section detailed 5 applications of probability and statistics to the practice of financial risk management in J P Morgan's RiskMetrics™. The statistical framework rests on the Gaussian approximation to the distribution of compound returns on financial instruments, including bonds, equities, currencies, and commodities. Market risk due to an adverse market move in the price of the underlying instrument then has a natural calibration in terms of the standard deviation or volatility of the historical compound returns. These volatilities are used to calibrate the value-at-risk or VaR in a trading position in the underlying security. For portfolios of securities, the multivariate generalization of value-at-risk is the daily-earnings-at-risk or DEaR, which calibration requires the historical correlations among the instruments in the portfolio. The so-called additive case gives a conservative valuation of risk recommended by the Basel Committee in which DEaR is the sum of the VaRs.

For the complement of instruments monitored in The RiskMetrics™Data set, the market

[4]James Kuczmarski, personal communication.

moves are represented in terms of locally Gaussian processes or time series whose local characteristics specify the infinitesimal mean and the infinitesimal covariance of short term compound returns. The span of a risk management system is then defined in terms of the number of sources of uncertainty in financial markets that can be covered by the complement of monitored instruments. That the system spans less than all sources of uncertainty is one reason for the need to map certain instruments to a standard set such as the synthetic government zeroes.

One of the critical features of The RiskMetrics™Methodology is the practice of single exponential smoothing of the (squared) market innovations for estimating volatilities and correlations from historical returns. The rationale is to capture the short term behavior of market volatility as modeled by the widely successful stochastic volatility models; one model justifies the exponential filters as the optimal one-step-ahead forecast of volatility. Exponential smoothing has the virtue of responding quickly to punctuated, sharp movements in market volatility, while attenuating their long term effects at a geometric rate.

Modern pricing theory is based on functionals of Itô processes and partial differential equations, so market risk for options is often complicated by nonlinear dependence of the option price on characteristics of the underlying security. Although the parametric approach to valuing risk based on the Taylor series expansion or the "greeks" works well enough for some situations, complicated options require full evaluation based on a computationally intensive scenario approach using structured Monte Carlo simulations.

Finally, mapping securities involves a three step algorithm for identifying cash-flow equivalents among the securities monitored in The RiskMetrics™Data set. Equities map to market indices, bonds to synthetic zeroes, and so on. The case of the synthetic government zero yield curve was treated in some detail as an application of the general theory of Markov processes and the principle of penalized least squares. The penalty arose from a smoothness criterion or goodness-of-fit for the forward rates to the prior, mean-reverting forward rates. The prior rates were derived from an underlying Itô process for the term structure of interest rates. Although we did not discuss mapping for commodity price risk here, there is an analogous theory for commodity risk based on a term structure drawn from futures contracts.

3. DIRECTIONS OF RESEARCH

This section is an outline of directions of research on applications of probability and statistics to the practice of financial risk management. The directions outlined here lie along lines of research in data analysis for multivariate dependencies, models on Markov processes, and inference for stochastic processes, but there is enough detail in the discussion above for each interested reader to do the same for other areas of statistical expertise.

The quality of the distribution theory for compound returns varies with the asset class and the part of the distribution being described. Returns of foreign exchange trading conform better than interest rates, the middle of the distributions better than the tails. This is an instance of Winsor's Principle which Tukey (1962) underscored to signal the tendency for much empirical data to appear Gaussian in the middle but not in the tails. The heavier tails of compound returns challenges the value-at-risk as a measure of market risk, because it anticipates little about the probability nor the size of volatile market moves. The theory of modeling and inference with heavy tailed time series may provide feasible procedures for anticipating such risk.

The analysis of multivariate returns appears limited to univariate and bivariate marginal distributions, not enough to thoroughly understand risk in portfolio positions. This is a ripe area for dedicated applications of high-dimensional exploratory analyses based on variance components, transition coplots, and graphical models for multivariate dependencies. The effort might accurately quantify, for example, the empirical observation that big market moves tend to reduce (or nearly so) the rank of the underlying covariance. Thresholding on correlations will show that such conditions further reduce the effective span of the risk management system.

The introduction of Engle's (1982) stochastic volatility models brought about a conceptual shift in modeling financial time series. These two dimensional models have an analogue in the phase space models used by engineers to analyze the state and velocity of dynamical systems. In his study of the Poisson clumping heuristic, Aldous (1989) called on the engineer's perspective in the context of the study of extremes for stationary processes. Calculations like Rice's formula suggest ways of moving valuations of market risk beyond the confines of Winsor's Principle. And beyond the pervasive application of the heirs of ARCH, this analytical framework promotes new insights particularly in the context of Lévy processes and hidden Markov models for switching regimes.

The theory of Zakai equations for optimal nonlinear filtering has played an important role in applications of hidden Markov models to signal processing, particularly when coupled with Girsanov's transformation in the construction and simulation of state estimation procedures. Although hidden Markov models have been applied to financial time series, the full weight of this theory has yet to be brought to bear on the problem of forecasting volatility breaks in realistic financial models. This analytical paradigm is also suitable to the study of models for the term structure of interest rates with the goal of filtering bond prices for forward rates of interest. The general theory is well suited to the study of more widely supposed but sophisticated models of term structure such as the one developed by Heath, Jarrow and Morton (1992). And because of the essentiality of scenario approaches to risk in options and other instruments, the paradigm also provides a framework for the design and implementation of simulation schemes.

In addition to these problems of modeling, filtering and simulation for stochastic processes, the discussion of the previous section suggests problems of parametric estimation, nonstationarity, heterogeneity, and high-dimensional data visualization for further consideration. Without significant addition of computational burden, there are a number of specific instances in which formal procedures might replace adhoc ones for fixing external parameters or dealing with nuisance parameters.

Finally, graphical representations of risk in portfolios need to meet the exigencies of traders and managers. The market already has one product for doing so in real time in 3 dimensions that can be studied in the context of the underlying statistical objectives of the risk management system. For the industry, there is presently no comparative theory of risk management systems that can quantify the relative benefits to firms and evaluate recommendations.

5. CONCLUSION

This case study focused on 5 applications of probability and statistics to the practice of financial risk management in J P Morgan's RiskMetrics™. In outlining the fundamental practices and principles of the methodology, the discussion provides both an introduction to RiskMetrics™and a platform for further collaborative research on statistics in risk management systems. Parts of the document are suitable for discussion in statistics courses with masters students of business administration, other parts for consideration with doctoral students.

The framework for studies of risk management in Oldfield and Santomero (1994) divides the landscape of risk management research into three interlocking areas of research: 1) market risk, regulation and other environmental factors, 2) credit risk, operational risk and other specific types of risk, and 3) valuation and aggregation of different types of risk for firm-level risk management. Since risk management systems cut across these principal areas, continuing research on the functions of systems like RiskMetrics™will provide theoretical insights and a practical basis for understanding some of the unifying themes.

*   *   *

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APPENDIX A

INTRODUCTION TO RISKMETRICS™

Morgan Guaranty Trust Company

Third Edition, May 1995

New York
November 21, 1995

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JPMorgan

Introduction to RiskMetrics™

Fourth edition

- **A methodology to estimate market risk based on the Value-at-Risk approach**
- **A set of consistently calculated volatilities and correlation forecasts for use as inputs to estimate market risks**
- **A methodology and data engine for risk management systems developed by J.P. Morgan and third parties**

Market risk has become one of the most significant concerns of participants in the financial markets. Regulatory agencies, commercial and investment banks, corporates, and institutional investors are all increasingly focusing on the level of market risk incurred by their institutions. Because of the increased attention to risk, in October 1994, J.P. Morgan released **RiskMetrics™**, a market risk estimation methodology which builds on Morgan's market risk management experience, accompanied by volatility and correlation datasets covering the major financial markets.

Our motivation for promoting RiskMetrics™ is threefold:

1. We are interested in promoting greater **transparency** of market risks. Transparency is the key to risk management.
2. We want to provide a **benchmark** for market risk measurement to allow comparison of risks. Risks can only be compared when they are measured with the same yardstick.
3. We are making **sound advice** available to our clients on managing their market risks. We describe the RiskMetrics™ methodology as the basis to understanding and evaluating risk management techniques.

RiskMetrics™ is based on, but differs significantly from, the system developed by J.P. Morgan for the measurement, management, and control of market risks in its trading, arbitrage, and own account investment activities. **We remind our readers that no amount of sophisticated analytics will replace experience and professional judgment in managing risks.** RiskMetrics™ should thus be seen as nothing more than a high-quality toolbox for the professional risk manager involved in financial markets and is not a guarantee of specific results.

Why the current interest in discussing methodology and implementation of market risk monitoring?

The answer lies in the significant changes that the financial markets have undergone over the last two decades. **First**, global securities markets have expanded and both exchange-traded and over-the-counter derivatives have become a major component of the markets. In addition, new instruments such as FRA's have been introduced, made more complex, or their usage extended, e.g., commodity swaps, and there is more active balance sheet management.

These developments, along with advances in computer technology, have gone hand in hand with changes in management practices. Many firms are moving away from management based on accrual accounting and towards risk management based on marking-to-market positions. Increased liquidity, pricing availability and trading focus has led to the implementation of frequent revaluation of positions or the mark-to-market concept.

As balance sheets have become more liquid, the potential for frequent and accurate reporting of investment gains and losses has led an increasing number of firms to either manage or account for daily earnings from a mark-to-market perspective. The switch from accrual accounting to mark-to-market resulted in higher swings in reported returns, therefore increasing the need for managers to focus on the volatility of the underlying markets. The markets have not suddenly become more volatile, but the focus on risks through mark-to-market has highlighted the potential volatility of earnings.

Given the move to frequent revaluation, managers have become concerned with estimating the effect of changes in market conditions on the value of their positions.

Second, significant efforts have been put into developing methodologies and systems to measure investment performance. Indices for foreign exchange, fixed income securities, and equities have become commonplace and are used extensively to monitor returns, compare assets or asset classes, and allocate funds.

The somewhat exclusive focus on returns, however, has often led to incomplete performance analysis. Gross return measurement gives no indication of the costs in terms of

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risk or the estimated volatility of returns. It is commonly accepted that higher returns can only be obtained at the expense of higher risk or through diversification. While this trade-off is well known, the risk measurement component of its analysis has not yet received sufficient attention.

Market risk is just one form of risk to which participants are subject in the financial markets. Risk takes many forms and is often defined as the probability associated with a loss. The most common classifications of risk are based on the nature of the underlying uncertainty:

- **Credit risk** estimates potential losses due to the inability of a counterparty to meet its obligations.
- **Operational risk** results from the errors that can be made in instructing payments or settling transactions.
- **Liquidity risk** is associated with the inability of a firm to fund illiquid assets.
- **Market risk** involves the uncertainty of earnings resulting from changes in market conditions such as the asset prices, interest rates, volatility, and market liquidity. Market risk can be absolute or relative. **Absolute** market risk estimates a potential total loss expressed in currency terms, for example, Dollars at Risk. Trading managers focus on how much they can lose over a relatively short time horizon such as one day. This is called DEaR, Daily Earnings at Risk. In some cases, the investment horizon, or the time needed to unwind a position, is longer, such as a month. J.P. Morgan refers to this case as VaR, Value at Risk. **Relative** market risk measures the potential for under performance, i.e., estimated tracking error, against a benchmark. The investment management industry uses this version of market risk.

While most market participants have long focused on trying to quantify credit risks, very few institutions, even in the banking and securities sectors, have developed practical measures of aggregated market risk. Investors and trading managers are looking for common standards to measure market risks to better estimate the risk/return profile of individual assets, asset classes or entire firms. Notwithstanding the external constraints from the regulatory agencies, the management of financial firms has been looking for ways to measure the level of market risk incurred by their businesses given the potentially damaging effect of miscalculated risks on company earnings.

A common framework for measuring market risk has been lacking to date. While the financial industry has produced a wide variety of indices to measure return, it has done much less to facilitate the measurement of risk.

In response, J.P. Morgan has made the following products available:

1. **RiskMetrics™ VaR methodologies: a description of a series of market risk methodologies to map the cash flows of positions and estimate their market risk.** Value at Risk is an estimate, with a predefined confidence interval, of how much one can lose from holding a position over a set horizon. Potential horizons may be one day for typical trading activities or a month or longer for portfolio management. The methods described in our documentation use historical returns to forecast volatilities and correlations that are then used to estimate the market risk. These statistics can be applied across a set of asset classes covering products used by financial institutions, corporates, and institutional investors. RiskMetrics™ supports a series of VaR methodologies, from the "delta" valuation approach where changes in the value of a position are approximated by a linear function (VaR = value of position times price volatility of instrument) to a full simulation approach where all instruments are revalued under different scenarios. These alternatives for market risk estimation are described in the RiskMetrics™ Technical Document *(3rd edition, May 1995)*.

2. **RiskMetrics™ datasets: a comprehensive set of daily reestimated volatilities and correlations** across a large number of asset classes and instruments. The datasets are an important input to any risk management model. These datasets contain forecasts of financial asset volatilities and their correlations that can drive simulations of market risk. The methodology for estimating the volatilities (defined using a 95% confidence interval) and correlations is fully transparent and consistent across asset classes. Three datasets are currently available: one applicable for estimating risk over a 24 hour horizon, one designed for market participants with a 1-month horizon and the last developed to meet the requirements contained in the latest proposals from the Bank for International Settlements on the use of internal models to estimate market risk. The datasets are updated daily and distributed to the Internet and Compuserve.

Data structure

J.P. Morgan produces the three datasets of volatility and correlation estimates on over 400 instruments. In total, each dataset contains 450 volatilities and over 100,000 correlations.

RiskMetrics™ instruments and markets

	Foreign Exchange	Money Markets	Int. Rate Swaps	Gov't. Bonds	Equity Indices
Australia	x	x	x	x	x
Austria	x	x			x
Belgium	x	x	x	x	x
Canada	x	x	x	x	x
Denmark	x	x	x	x	x
Finland	x	x			x
France	x	x	x	x	x
Germany	x	x	x	x	x
Hong Kong	x	x	x		x
Ireland	x	x	x	x	x
Italy	x	x	x	x	x
Japan	x	x	x	x	x
Netherlands	x	x	x	x	x
New Zealand	x	x	x	x	x
Norway	x	x			x
Portugal	x	x			x
Singapore	x	x			x
Spain	x	x	x	x	x
Sweden	x	x	x	x	x
Switzerland	x	x	x		x
U.K.	x	x	x	x	x
U.S.	x	x	x	x	x
ECU	x	x	x	x	

The RiskMetrics™ datasets cover foreign exchange, money markets, interest rate swaps, bonds, and equity indices in 23 countries, along with commodities.

Complete term structure data, varying according to currency, is available for most fixed income instruments. Commodity term structure data is provided using liquid futures and spot data where available. Energy commodities include West Texas Intermediate (WTI), Heating oil, Unleaded gas, and Natural gas. Metals include Aluminum, Copper, Nickel, and Zinc. Precious metals include Gold, Silver, and Platinum.

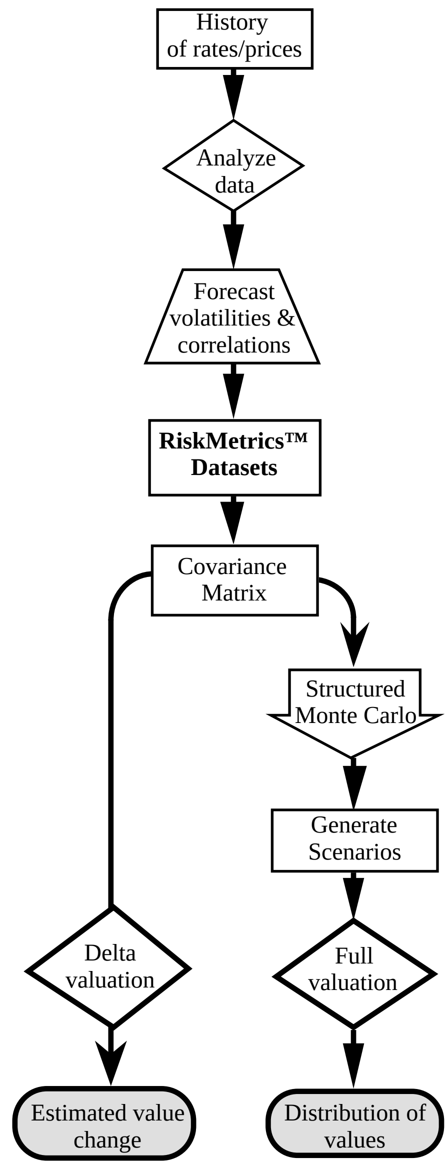
For a detailed description of the data structure, please refer to the *RiskMetrics™ Technical Document.*

Implementing market risk management

At all levels of the risk management process, be they individual position (micro), portfolio (macro) or global business lines (strategic), the risk management process should meet a number of attributes:

- **Transparency of risk** is of paramount importance because it is often the unforeseen risks which cause the biggest problems. Risks will not be properly managed if they are not identified.

Use of the RiskMetrics™ datasets for VaR estimation



- **Rigorous risk measurement techniques** are the “science” portion of the discipline. The theoretical precepts behind Value at Risk are not new. The theory is based on the standard error of a Normal distribution as is Markowitz's portfolio optimization techniques from the 1950s, which have been applied to investment management for some time. Most of the recent advances have come from understanding how to apply well-known techniques to new areas such as derivatives risk management.

- **Timely, quality information** requires a significant investment in systems architecture to pull together all risk management information together at the corporate level.

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- **Diversification of risks** is the goal of a good risk management process. In addition, hidden concentrations of exposure to a single counterparty which represent under-priced risks need to be uncovered.

- **Independent oversight** of risk taking activities is now viewed as a must by practitioners and regulators.

- **Use of disciplined judgment** is the "art" aspect of the business which relies on a sound understanding of how to use risk management tools and their limitations.

Implementing a risk management process which meets these criteria often requires significant effort on the part of market participants. With RiskMetrics™, we have attempted to commoditize some of the tools required to meet the requirements listed above.

There are two major requirements to setting up a risk management framework, such as RiskMetrics™, to estimate market risks:

- **Quality data** must serve as the basis for estimating sound statistics of future market movements, i.e., volatilities and correlations. Knowledge of the data's properties is paramount and often sheds light on the reliability and performance of market risk estimates. Within the RiskMetrics™ framework, procedures have been implemented to address a number of common data problems. One is the distinction between multiple outliers and influential observations. In other cases, data may not be available because of market closures. Furthermore, data that may include prices and rates recorded at different times, i.e., non synchronous data, may cause covariances to be underestimated. Nevertheless, variance-covariance estimates may be numerically unstable.

- **A comprehensive mapping system** must represent positions in a consistent manner. Although seemingly simple, most practitioners know that the logistical problems of collecting accurate position data within an institution may be overwhelming. The first problem is to obtain accurate position data across different business areas. The second is to agree on a methodology to map positions consistently. In the fixed income world alone, there are various ways to describe the same position and its exposure to risk. A portfolio of bonds may be described in terms of its duration. This concept often is used to estimate risk but suffers from the fact that it only measures changes in value resulting from small parallel yield curve shifts. Positions may also be described as a stream of time dated principal flows. However, that approach does not correctly estimate the

(continued on page 6)

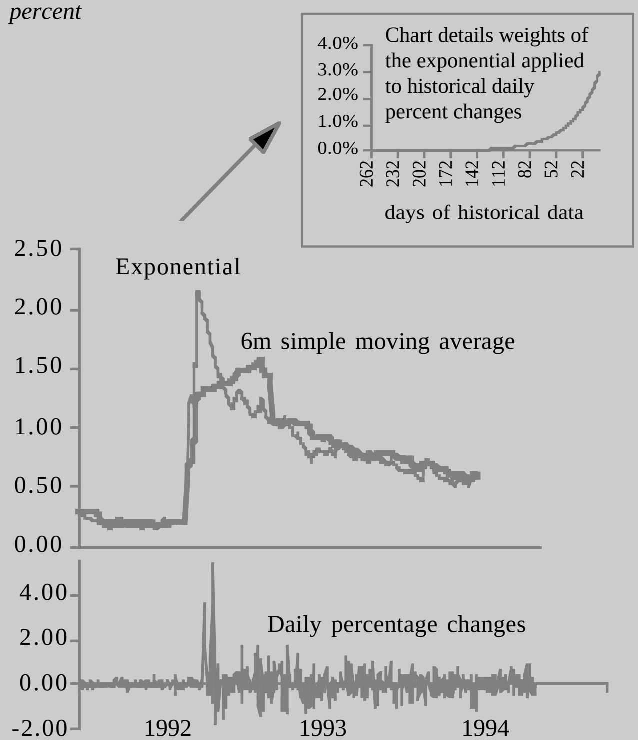
Measures of volatility and correlation

Traditional practice has been to estimate current volatility in the financial markets as the standard deviation of changes in price or yield over a set prior period, such as six months or a year. This approach explicitly allocates an identical weight to all of the observations and leads to volatility estimates that can decline abruptly once a large influential observation falls out of the measurement sample.

The RiskMetrics™ approach uses exponential moving averages of historical rate and price returns to forecast future volatility in order **to ensure responsiveness to market shocks and a subsequent gradual decline in the forecast of volatility**.

The chart below demonstrates the differences between the two approaches as they estimated volatility during the ERM crisis of September 1992 for the Lira/DM exchange rate. The exponential moving average shot up rapidly, adjusting for the devaluation of the Lira but then started to decline. **The simple 6-month moving average measure continued to rise until early 1993 and then fell by 30% in April of that year**. When compared to the actual daily changes plotted below, the exponential offers a better representation of actual volatility.

Lira/DM daily volatility
percent



impact on risk of coupon flows when they are off current market rates. The method outlined and recommended in the *RiskMetrics™ Technical Document* is to decompose all fixed income instruments into their component cash flows and apply estimated volatilities of zero coupon rates to each individual cash flow.

The methodology outlined in the *RiskMetrics™ Technical Document* relies on a number of assumptions that must be clearly understood in order to interpret results with an objective view. Market risk measurement is as much an art as it is a science and the potential and limits of the methodology must clearly be understood.

The RiskMetrics™ methodology uses **historical return data** to forecast how the markets are likely to move in the future over a specified horizon. This is a methodological choice. There are various alternative approaches to forecasting future volatility. One is the internal forecast method where market risk professionals are asked for their estimates. The problem with this method is that it is subjective and cannot be practically implemented for a large dataset. A second method is to use implied volatilities and correlations extracted from options prices. The problem with this approach is that quality data is difficult to obtain for a wide range of rates. Good data only exist for derivatives traded on established exchanges. We, therefore, base RiskMetrics™ volatility and correlation forecasts on recently observed price and rate return histories.

The estimates of volatilities and correlations that comprise the RiskMetrics™ dataset assume that changes in prices and yields of financial instruments are **normally distributed**. Given this assumption, volatility is expressed in terms of standard deviation. The RiskMetrics™ approach has been to use 1.65 standard deviations as its measure of risk which encompasses 95% of occurrences. The assumption has two important implications:

- Occurrences outside the 95% confidence interval theoretically occur 1 day out of 20. Therefore, the estimates of volatility will underestimate risk one day a month by an unspecified amount. The simulation of worse case scenario impacts on the value of positions cannot be easily implemented using standard probability distributions.

  Stress testing scenarios must be run using various assumptions in order to provide risk managers with insights into deviations from normality and cases of event risk.

- Most return distributions have fat tails. RiskMetrics™ takes this into account by allowing volatilities to change daily and using exponential weighting.

The examples on page 7 shows how to estimate the market risk of a simple set of fixed income positions using forecast volatilities and correlations.

Practical uses of market risk information

Estimating the amount of market risk taken by an institution can serve a number of purposes:

- **Management information.** Senior management is informed of the risks run by the trading and investment operations of the institution. Ideally such information is an integral part of a comprehensive management information system which also covers areas such as credit and operational risk.

- **Setting of limits.** Position limits have traditionally been expressed in nominal terms, futures equivalents, or other denominators unrelated to the amount of risk effectively incurred. Setting limits in terms of Value at Risk has significant advantages. For example, position limits become a function of risk and positions in different markets or products can be compared through a common measure.

- **Resource allocation.** Using Value at Risk information, risk-takers can make more informed decisions about their trading strategies. From a tactical point of view, positions may be taken which maximize the return over risk potential. Strategically, profit objectives across businesses can become a function of the risk incurred. Management can use profit to risk ratios to allocate resources to specific businesses which offer more overall potential in terms of their risk/reward profile.

- **Performance evaluation.** To date, trading and position taking talent have been rewarded to a significant extent on the basis of total returns. Estimated and realized volatility of profits adds an extra dimension to performance evaluation. Ratios of P/L over risk (risk ratio) and of P/L over volatility (Sharpe Ratio) can be combined into what we would define as a trader's efficiency ratio (estimated risk/realized volatility) which measures an individual's capacity to translate estimated risk into low realized volatility of revenues.

- **Regulatory reporting.** Financial institutions such as banks and investment firms will soon have to meet capital requirements to cover the market risks that they incur as a result of their normal operations. The Basel Committee of the BIS has presented proposals to both estimate market risk and define the resulting capital requirements to be implemented in the banking sector. The European Union has approved a directive (EEC 93/6), effective January

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1996, that mandates banks and investment firms to set capital aside to cover market risks. Both of these proposals have been the object of heated debates among practitioners. In the United States, the Securities and Exchange Commission is considering imposing market risk disclosure requirements to all entities who file financial statements.

While the latest proposals from the BIS have gone a long way in addressing practitioners' concerns, a number of issues remain unresolved. The BIS proposals will allow banks to use internal models to estimate market risk, but they will also impose stringent quantitative requirements on some of the factors used in these models. First of all, the regulatory framework proposed by the BIS does not reward diversification strategies to any significant extent. Correlations can be applied within but not across asset classes to reduce risk estimates. Furthermore, the BIS has arbitrarily set certain parameters (length of historical window to measure volatility, multiplier between value at risk estimate and capital allocation, and choice of 10 day risk horizon) without any known methodological justification. As a result, the strict application of the current recommendations could lead financial institutions to overestimate market risk and subsequently be overcapitalized.

Where to get the RiskMetrics™ datasets

The RiskMetrics™ dataset is available daily by 10:30 a.m. U.S. Eastern Standard Time (based on the previous day's market close) on a number of systems which include:

- **Internet**: RiskMetrics™ publications and datasets are posted daily on a J.P. Morgan server accessible through the Internet. Users wishing to browse through the Web can use Mosaic or other equivalent browsers (URL http://www.jpmorgan.com). Files can also be accessed and downloaded via anonymous ftp.

- **CompuServe®**: Users can access the J.P. Morgan forum on the CompuServe® Information Service from around the globe, generally via a local phone call. To download RiskMetrics™ datasets and publications, type "go jpm-14" at the prompt. The RiskMetrics™ section of the J.P. Morgan forum can be accessed without a user id and password.

Subsets of the volatility and correlation data are distributed on Reuters (RKMS-Z), Telerate (17379-17385), and Bloomberg (RMMX Go).

RiskMetrics™ related products

A methodology and the underlying market data are not sufficient to enable users to implement internal market risk management systems. Therefore, J.P. Morgan has encouraged third-party consultants and software developers who are committed to developing risk management estimation and reporting tools to utilize the RiskMetrics™ methodology and data. Third parties have strongly endorsed RiskMetrics™ by incorporating it into new or existing systems. Their products are geared to providing participants in the financial markets with the tools necessary to estimate the risks resulting from exposure to market movements. For a detailed description of the various consultants and software developers who have applied RiskMetrics™ to risk management systems and other products, please refer to the *RiskMetrics™ Directory (available on-line from http://www.jpmorgan.com/MarketDataInd/RiskMetrics/Third_party_directory.html*).

Example 1: A single position Value at Risk example

Definitions

Value at Risk = the forecasted amount that may be lost given an adverse market move
= Amount of Position * Volatility of Instrument

Volatility = % of value which may be lost with a certain probability, e.g. 95%

Position

A U.S. investor is long 140 million Deutsche marks

Market/Risk Information

DEM/USD FX Volatility:	0.932%
FX Rate:	1.40 DM/USD

Value at Risk

$\mathrm{VaR_{USD}}$ = DEM 140 million * 0.932% / 1.40 DEM/USD = USD 932,000

Example 2: Two position Value at Risk example

Definitions

$$VaR = \sqrt{VaR_1^2 + VaR_2^2 + 2\rho_{12} VaR_1 VaR_2}$$

VaR 1 = Value at Risk for Instrument 1
VaR 2 = Value at Risk for Instrument 2
ρ_{12} = Correlation between the price movements of Instrument 1 and 2

Position

A U.S. investor is long DEM140 million 10-year German bunds
(the U.S. investor is therefore also long Deutsche marks)

Market/Risk Information

Bund volatility:	0.999%
DEM/USD FX Volatility:	0.932%
Correlation:	-0.27

Interest rate risk	DEM 140 million * 0.999% / 1.40 = USD999,000
FX risk	DEM 140 million * 0.932% / 1.40 = USD932,000

Value at Risk

$$\mathrm{VaR_{USD}} = \sqrt{(999,000)^2 + (932,000)^2 + 2 * (-0.27) * 999,000 * 932,000} = \text{USD 1.17 million}$$

The undiversified risk, assuming a perfect correlation between instruments, is simply the sum of the individual risks. For example with a +1.0 correlation this would be USD999,000 + USD932,000 = USD1.93 million. We look at the undiversified risk with respect to the diversified risk (our typical VaR calculation). The difference between the two is the diversification benefit to the investor due to correlation ($1.93 million - $1.17 million = $760,000).

For N positions

$$DEaR = \sqrt{\vec{V} * [C] * \vec{V}^T} \quad \text{where}$$

$$\vec{V} = \begin{bmatrix} DEaR_1 & \ldots\ldots & DEaR_n \end{bmatrix} \quad \text{(DEaR vector of individual positions)}$$

$$[\mathrm{C}] = \begin{bmatrix} 1 & .. & \rho_{n1} \\ .. & 1 & .. \\ \rho_{1n} & .. & 1 \end{bmatrix} \quad \text{(correlation matrix)}$$

$$\vec{V}^T = \begin{bmatrix} DEaR_1 \\ \ldots\ldots \\ DEaR_n \end{bmatrix} \quad \text{(transposed vector of V)}$$

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RiskMetrics™ data and publications

Introduction to RiskMetrics™: An eight-page document which broadly describes the RiskMetrics™ methodology for estimating market risks.

RiskMetrics™ Technical Document: A 200-page manual describing the RiskMetrics™ methodology for estimating market risks. It specifies how financial instruments should be mapped and describes how volatilities and correlations are estimated in order to compute market risks for trading and investment horizons. The manual also describes the format of the volatility and correlation data and the sources from which daily updates can be downloaded.

RiskMetrics™ Directory: Available exclusively on the Internet, a list of consulting practices and software products that incorporate the RiskMetrics™ methodology and datasets.

RiskMetrics™ Monitor: A quarterly publication which discusses broad market risk management issues, statistical questions as well as new software products built by third-party vendors that incorporate RiskMetrics™.

RiskMetrics™ datasets: Two sets of daily estimates of future volatilities and correlations of approximately 450 rates and prices – each a total of 100,000+ data points. One set is for short-term trading risks, the other for medium-term investment risks. Datasets currently cover Foreign Exchange, Government Bond, Swap, and Equity markets in up to 22 currencies. Eleven commodities are also included. A RiskMetrics™ Regulatory dataset which incorporates the latest recommendations from the Basel Committee on the use of internal models to measure market risk is also available.

Bond Index Cash Flow Maps: A monthly insert into the Government Bond Index Monitor outlining cash flow maps of J.P. Morgan's bond indices. Available on the Internet.

Trouble accessing the Internet? If you encounter any difficulties in either accessing the J.P. Morgan home page on http://www.jpmorgan.com or downloading the RiskMetrics™ data files, you can call 1-800-JPM-INET in the United States.

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APPENDIX B
GLOSSARY OF RISK MANAGEMENT

Absolute Market Risk: Risk associated with the change in value of a position or a portfolio resulting from changes in market conditions such as yields or prices.

Adverse Market Move: Defined in RiskMetrics™ as 1.65 times the standard error of returns and the basis for calculations of Value-at-Risk.

Beta: A volatility measure and the slope of the regression of the return on an equity on the return on a market porfolio, dependending naturally on the covariance between an equities return and the return on the market portfolio.

CAPM: Capital Asset Pricing Model. A model which relates the expected return on an asset to the expected return on the market portfolio.

Constant Maturity: The fixing of maturities on a time series of bonds when accounting for bonds "rolling down" the yield curve.

Decision Horizon: The time period between entering and unwinding or revaluing a position. Currently, RiskMetrics™ offers statistics for 1-day and 1-month horizons.

Decay Factor: The discount in exponential smoothing of time series, see lambda.

Delta Equivalent Cash Flow: In situations when the underlying cash flows are uncertain (e.g. option), the delta equivalent cash flow is defined as the change in an instrument's fair market value with respect to its discount factor. These cash flows are used to find the net present value of an instrument.

Delta Neutral Cash Flows: These are cash flows that exactly replicate a callable bond's sensitivity to shifts in the yield curve. A single delta neutral cash flow is the change in the price of the callable bond divided by the change in the value of the discount factor.

Duration (Macaulay): Duration is the derivative of bond price with respect to rate of interest, which can be written in terms of a weighted average term of a security's cash flow.

GAAP: Generally Accepted Accounting Principles.

Historical Simulation: The application of past market data to study the propoerties of a present portfolio.

λ **lambda (decay factor):** The symbol used in RiskMetrics™for the discount coefficeint in single exponential smoothing. In RiskMetrics™ $\lambda = 0.94$ for calculating volatilities and correlations over 1–day horizons and $\lambda = 0.97$ over 1–month horizons.

Linear Risk (nonlinear): For a given portfolio, the payoff is linear in the underlying prices and rates. Otherwise, the risk is said to be nonlinear, the usual case with options.

Compound Return: For price or rate P_t and P_{t+1}, the one-period compound return is $\ln(P_{t+1}/P_t)$, whereas the return is $(P_{t+1} - P_t)/P_t$. For small returns, compound returns are well approximated by returns. These are usually reported as percentages, so multiplied by 100.

Mapping: The identification of the cash flow of actual positions with its cash flow equivalents in some standardized set of instruments. Duration, Principal, and cash flow.

Modified Duration: An indication of price sensitivity. It is equal to a security's Macaulay duration divided by one plus the yield, for a zero coupon bond with maturity T and yield Y, the modified duration is simply $T/(1+Y)$.

Relative Market Risk: Risk measured relative to an index or benchmark.

Residual Risk: The risk in a position that is issue specific.

Speed of Adjustment: For mean reverting time series, the speed of adjustment is 1 minus the coefficient of autoregression. For forward rates of interest, it is estimated from past data on short rates. A fast speed of adjustment will result in a forward curve that approaches the long-run rate at a relatively short maturity.

Stochastic Volatility: Models for financial time series that take the underlying variance of the process to be a stochastic process.

Structured Monte Carlo: RiskMetrics™term for using its estimated volatilities and correlations to to generate Gaussian variables for the purpose of simulating future price scenarios.

Total Variance: The variance of the market portfolio plus the variance of the return on an individual asset.

APPENDIX C

GLOSSARY OF STATISTICS

ARCH: Autoregressive Conditional Heteroskedascticity. A time series model of volatility or scale of returns as dependent on past returns. GARCH – Generalized ARCH, models volatility as a function of past returns **and** past values of volatility. EGARCH – Exponential GARCH, IGARCH – Integrated GARCH, SWARCH – Switching Regime ARCH.

Autocorrelation (Serial Correlation): For time series, the notion of correlation over time.

Bootstrap: A resampling method for generating random samples with replacement from observed data for the purpose of studying the distribution of statistically unknown functionals of the underlying law. It is used in RiskMetrics™as a way to scale measures of volatility drawn from daily returns to those drawn from monthly returns.

Cholesky Decomposition: A standard decomposition of a symmetric positive definite matrix into the product of a lower triangular matrix and its transpose. The decomposition is used in RiskMetrics™structured Monte Carlo method for stimulation of multivariate normal returns.

EM Algorithm: A statistical procedure for the estimation of parameters in the presence of incomplete or missing data. EM stands for Expectation Maximization in that missing values are replaced by their expected values given the observed data before the maximization step.

Exponential Moving Average: For a time series, an average in which observations are weighted geometrically with respect to age.

Itô Process: A class of stochastic processes defined by an Itô stochastic differential equation with general coefficients and canonical driving processes given by the Wiener process and a Poisson random measure.

Kurtosis: Characterizes relative peakedness or flatness of a distribution: the sample kurtosis K is given by the formula

$$K = \left\{ \frac{N^2 - 2N + 3}{(N-1)(N-2)(N-3)} \sum_{i=1}^{N} \left(\frac{X_i - \bar{x}}{s_x} \right)^4 \right\} - 3\frac{(N-1)(2N-3)}{N(N-2)(N-3)}$$

Excess kurtosis is defined by the quantity $K - 3$, since the Gaussian distribution has a kurtosis of 3.

Leptokurtosis: The condition of a distribution having heavier tails than that of the Gaussian distribution.

(Sample) Mean: A measure of location or average, simply the sum of daily changes divided by the count

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

Mean Reversion: The tendency of some times series to revert by regression to some long run average value. The series then exhibits a quasi-cyclical behavior with respect to its mean, much like to behavior of an Ornstein-Uhlenbeck process or a stationary AR(1) process.

Mean Reversion Time: For mean reverting time series, the mean reversion time is the reciprocal of 1 minus the autoregression coefficient. Roughly speaking, the average duration of an excursion from the mean.

Parametric Model: When the functional form for the distribution of a random variable is assumed to depend on a finite number of real parameters. The mean and scale are the parameters of the Gaussian law.

Skewness: Characterizes the degree of asymmetry of the a unimodal distribution about its mean. Positive skew means long tailed to the right of the mean, negative to the left of the mean. The sample skew S is given by the formula

$$S = \frac{N}{(N-1)(N-2)} \sum_{i=1}^{N} \left(\frac{X_i - \bar{x}}{s_x} \right)^3$$

Standard Deviation: The scale of variability of a random variable. The sample standard deviation s_x is given by the formula

$$s_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{x})^2}.$$

Wiener Process: A square integrable martingale having initial value 0 and quadratic variation t. A standard Brownian motion, meaning a continous process having stationary and independent increments.