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Extensions of Modified DEA

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Extensions of Modified DEA [1]

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Abstract: Andersen and Petersen (1993) presented an extension of the basic DEA methodology, called *modified DEA*, which has the desirable feature of ranking not only the inefficient DMUs, but the efficient ones as well. However, when their basic approach is extended to the cases of variable returns to scale and non-discretionary inputs, several conceptual problems arise. This paper addresses these problems, and illustrates the proposed extensions to the modified DEA method using data from a major U.S. bank.

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1. Introduction

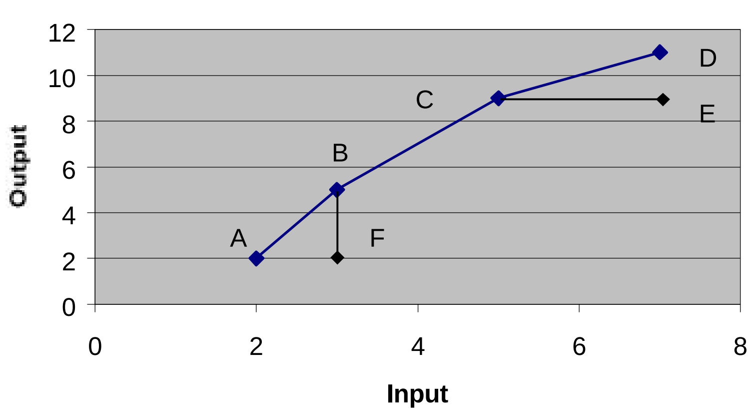
In Andersen and Petersen (1993), the Modified Data Envelopment Analysis method (MDEA) is presented using an input-oriented constant returns-to-scale (CRS) primal model. The core idea of MDEA is to exclude the decision-making unit (DMU) from the reference set while its own relative efficiency is being evaluated. MDEA has an advantage over the conventional DEA method in that MDEA provides a ranking procedure for all DMUs, including the efficient ones. However, there has been very little effort given to the development of MDEA models under different model orientations and returns-to-scale conditions.

Andersen and Petersen (1993) state that the same MDEA "… is applicable under conditions of non-increasing returns to scale (NIRS) and varying returns to scale (VRS)." However, after applying MDEA to a VRS model, it is possible for the primal model to be infeasible. Table 1 and Figure 1 present a simple example that illustrates the problems with the basic MDEA model in these situations.

Table 1: Example of MDEA under Variable Returns to Scale

DMU	Input	Output
A	2	2
B	3	5
C	5	9
D	7	11
E	7	9
F	3	2

Figure 1: Example of MDEA under VRS



Before we start to discuss this example, let us first define two terms for the remainder of the paper: *efficiency surface (frontier)* and *projection surface (frontier)* or *new efficiency surface (frontier*). In this paper, we will call the surface (frontier) consisting of the efficient DMUs identified by the conventional DEA models as the *efficiency surface (frontier)*, and the surface (frontier) consisting of the efficient DMUs identified by the MDEA model as the *projection surface (frontier)* or the *new efficiency surface (frontier*).

In this example, we have a single input X and a single output Y. Under variable returns to scale conditions, DMUs A, B, C, and D are efficient, and DMUs E and F are inefficient. For DMUs B and C, the MDEA method enables us to measure their efficiency with efficiency scores not necessarily equal to one. However, for DMU A and DMU D, the primal problems under certain model orientation become infeasible after applying MDEA method:

1. Assume that the model orientation is input-oriented. For DMU D, after it is excluded from the reference set as in the proposed MDEA method, the projection surface (new efficiency surface) for D consists of A-B-C-E. Since the output of E is the same as that of C, the segment C-E is parallel to the input axis. Thus, D will not be able to project onto the projection surface along the direction of increasing its input. In other words, DMU D can increase its input proportionately to positive infinity while remaining efficient, which results in its efficiency score going to positive infinity. Since the objective of the input-oriented primal model is to minimize its efficiency score, the primal problem for DMU D in the MDEA model is infeasible.
2. Assume that the model orientation as output-oriented. For DMU A, after it is excluded from the reference set as in the proposed MDEA method, the projection surface (new efficiency surface) for A consists of F-B-C-D. Since the input of F is the same as that of B, the segment F-B is parallel to the output axis. Thus, A will not be able to project onto the projection surface along the direction of decreasing its output. In other words, DMU A can decrease its output proportionately to negative infinity while remaining efficient, which results in its efficiency score going to negative infinity. Since the objective of the output-oriented primal model is to maximize its efficiency score, the primal problem for DMU A in the MDEA model is infeasible.

Therefore, by excluding the observation DMU from the reference set, MDEA makes it possible for the primal problem to be infeasible. Because the primal problems for some DMUs are infeasible, we are not able to assign any efficiency scores to those DMUs. Therefore, the ranking of the whole DMU set does not exist. Thus, the extension of the MDEA method to variable returns to scale can result in an inability to rank all DMUs.

Are situations like that described above rare? Unfortunately, the answer is no. In fact, the problem of some DMUs' primal problems becoming infeasible in a variable returns-to-scale (VRS) model is unavoidable unless one of the following two situations holds:

1. In the input-oriented MDEA VRS model, if one DMU has the largest value of some component of the output vector, there exists at least one other DMU that has the same value for this component of the output vector. That is, for each component of the output vector, there exist at least two DMUs with the same maximal value.
2. In the output-oriented MDEA VRS model, if one DMU has the smallest value of some component of the input vector, there exists at least one other DMU that has the same value for this component of the input vector. That is, for each component of the input vector, there exist at least two DMUs with the same minimal value.

In practice, the possibility to get such a data set where these conditions would hold is very low. Thus, it is important to solve this infeasibility problem in order to apply the MDEA method in practice.

In this paper, we find that this problem can be dealt with through a redefinition and reinterpretation of the MDEA efficiency scores. In addition, based on this reinterpretation, we identify a special subset of the set of the strongly efficient DMUs, which we define as super efficient DMUs.

In practice, we also need to deal with non-discretionary inputs and outputs when applying a DEA model. Thus, it is important to evaluate the relative efficiency and estimate to what extent the discretionary inputs or outputs should be changed to make the inefficient DMUs efficient, while their non-discretionary inputs or outputs are "fixed" at their current level. In the literature, there has been some work done in dealing with non-discretionary inputs or outputs in conventional DEA models (Banker and Morey 1986; Golany and Roll 1993). However, there has been no effort given to the development of MDEA models with non-discretionary inputs or outputs. In order to explore the power of MDEA in practice, it is necessary and useful to develop such models as well. Also, we find that we cannot generally apply the current theorem about the relationship between the two groups of efficiency scores from the models with certain inputs as non-discretionary and the models with those inputs as discretionary to MDEA. It is also necessary to address this conceptual problem.

The remainder of the present paper is organized as follows. In Section 2, the formulations of the input-oriented MDEA primal and dual models with non-discretionary input under variable returns-to-scale conditions are presented. In Section 3, the definition and properties of super efficient DMUs are introduced and their properties are examined. In Section 4, we address another conceptual problem concerning the relationship between the two groups of efficiency scores from the models with some of the inputs as non-discretionary and the models with those inputs as discretionary in MDEA. In Section 5, an example using the data from a major U.S. bank is used to illustrate the proposed methodology. The main conclusions of this paper are summarized in Section 6.

2. Model Formulations

To begin, let us define the following notation:

I-----the set of all DMUs (decision making units),

M----the set of inputs,

D----the set of discretionary inputs,

F ----the set of non-discretionary inputs,

N-----the set of outputs,

ε----- small (non-Archimedean) positive number.

Here, $D \cup F = M$, and $D \cap F = \emptyset$.

The input-oriented VRS MDEA primal and dual models with non-discretionary inputs for DMU k are given as follows:

Model 1: Primal Model

$$Min \quad \theta_k - \varepsilon \cdot \sum_{n \in N} s_{kn}^{+} - \varepsilon \cdot \sum_{d \in D} s_{kd}^{-} \quad (1)$$

s.t.

$$\sum_{i \in I, i \neq k} \lambda_i Y_{in} - s_{kn}^{+} = Y_{kn} \;\; \forall\, n \in N \quad (2)$$

$$\theta_k X_{kd} - \sum_{i \in I, i \neq k} \lambda_i X_{id} - s_{kd}^{-} = 0 \;\; \forall\, d \in D \quad (3)$$

$$X_{kf} - \sum_{i \in I, i \neq k} \lambda_i X_{if} - s_{kf}^{-} = 0 \quad \forall\, f \in F \quad (4)$$

$$\sum_{i \in I, i \neq k} \lambda_i = 1 \quad (5)$$

$$\lambda_k = 0 \quad (6)$$

$$\lambda_i \geq 0 \;\; \forall\, i \in I \quad (7)$$

$$s_{kd}^{-} \geq 0 \;\; \forall\, d \in D \quad (8)$$

$$s_{kf}^{-} \geq 0 \;\; \forall\, f \in F \quad (9)$$

$$s_{kn}^{+} \geq 0 \;\; \forall\, n \in N \quad (10)$$

$$\theta_k \;\; free \quad (11)$$

Model 2: Dual Model

$$Max \quad \sum_{n \in N} \mu_n Y_{kn} - \sum_{f \in F} \nu_f X_{kf} + \mu_k \quad (12)$$

s.t.

$$\sum_{n \in N} \mu_n Y_{in} - \sum_{d \in D} \nu_d X_{id} - \sum_{f \in F} \nu_f X_{if} + \mu_k \leq 0 \;\; \forall\, i \in I,\, i \neq k \quad (13)$$

$$\sum_{d \in D} \nu_d X_{kd} = 1 \quad (14)$$

$$\mu_n \geq \varepsilon \;\; \forall\, n \in N \quad (15)$$

$$\nu_d \geq \varepsilon, \forall\, d \in D \quad (16)$$

$$\nu_f \geq \varepsilon, \forall\, f \in F \quad (17)$$

$$\mu_k \quad free \tag{18}$$

If F = Ø and M = D (i.e., there are no non-discretionary inputs in the model), then Model 1 and Model 2 can be simplified as follows:

Model 3: Primal Model

$$Min \qquad \theta_k - \varepsilon \sum_{n \in N} s_{kn}^{+} - \varepsilon \sum_{m \in M} s_{km}^{-} \tag{19}$$

s.t.

$$\sum_{i \in I, i \neq k} \lambda_i Y_{in} - s_{kn}^{+} = Y_{kn} \quad \forall\, n \in N \tag{20}$$

$$\theta_k X_{km} - \sum_{i \in I, i \neq k} \lambda_i X_{im} - s_{km}^{-} = 0 \quad \forall\, m \in M \tag{21}$$

$$\sum_{i \in I, i \neq k} \lambda_i = 1 \tag{22}$$

$$\lambda_k = 0 \tag{23}$$

$$\lambda_i \geq 0 \quad \forall\, i \in I \tag{24}$$

$$s_{km}^{-} \geq 0 \quad \forall\, m \in M \tag{25}$$

$$s_{kn}^{+} \geq 0 \quad \forall\, n \in N \tag{26}$$

$$\theta_k \quad free \tag{27}$$

Model 4: Dual Model

$$Max \qquad \sum_{n \in N} \mu_n Y_{kn} + \mu_k \tag{28}$$

s.t.

$$\sum_{n \in N} \mu_n Y_{in} - \sum_{m \in M} \nu_m X_{im} + \mu_k \leq 0 \quad \forall\, i \in I, i \neq k \tag{29}$$

$$\sum_{m \in M} \nu_m X_{km} = 1 \tag{30}$$

$$\mu_n \geq \varepsilon \quad \forall\, n \in N \tag{31}$$

$$\nu_m \geq \varepsilon \quad \forall\, m \in M \tag{32}$$

$$\mu_k \quad free \tag{33}$$

The formulations of the output-oriented MDEA VRS and the input-oriented MDEA CRS models can be formulated in an analogous manner.[1]

3. Super Efficient DMUs

3.1 Classification of Efficient DMUs in the Literature

In the literature, the traditional classification of efficient DMUs (Charnes, Cooper, and Thrall 1986; Seiford and Thrall 1990) partitions the efficient units into three subsets:

F------the set of the weakly efficient DMUs,

E'-----the set of the efficient DMUs, and

E------the set of the strongly efficient DMUs.

The efficiency of the DMUs in the three classes can be ranked from the highest to the lowest as follows: E>E'>F. Generally, DMUs belonging to E are the extreme points of the efficiency surface (frontier). DMUs belonging to F are the boundary points on the extended portion of the efficiency surface (frontier)[2]; i.e. the portion of the surface lying outside the convex hull of any subset of the strongly efficient DMUs (see, for example, Charnes, Cooper, and Thrall 1986). The rest of the boundary points on the efficiency surface (frontier) belong to E'. The efficient DMUs in E' can be expressed as the linear combination of other efficient DMUs while the strongly efficient DMUs in E cannot be expressed in this manner.

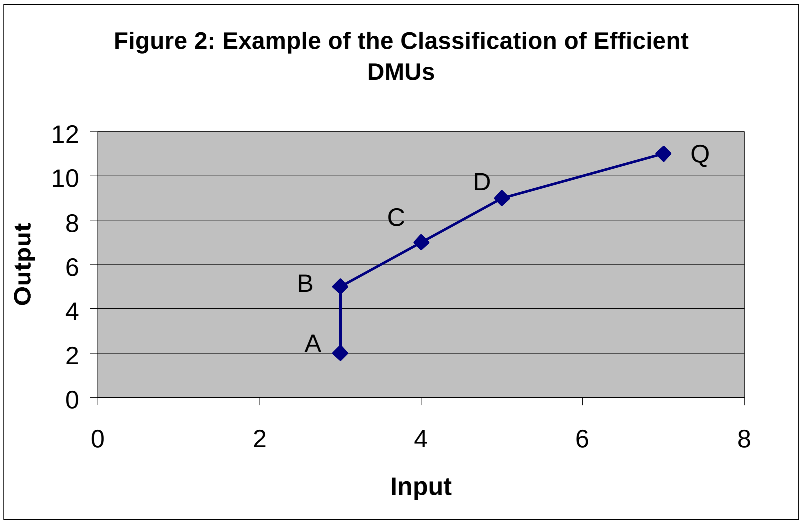
Table 2: Example of Classification of Efficient DMUs

DMU	Input	Output
A	3	2
B	3	5
C	4	7
D	5	9

[1] The formulations can be found on-line at the following address: http://opim.wharton.upenn.edu/~harker/MDEAAppendix.pdf

[2] In this paper, when we refer to "efficiency surface (frontier)", it includes the so-called "extended surface (frontier)" (Seiford and Thrall 1990).

Q	7	11



For example, in the set of the DMUs showed in Table 2 and Figure 2, DMUs A, B, C, D, and Q all have the efficiency scores equal to one in the input-oriented BCC model. DMU A (3, 2) is on the extended portion of the efficiency surface. Therefore, DMU A belongs to F. It is weakly efficient in the input-oriented model but not strictly efficient in the output-oriented model, since DMU B (3, 5) uses the same input as A but produces more output than A does. DMUs B, D and Q are all vertices of the efficiency surface and hence, DMUs B, D, and Q belong to E. None of DMUs B, D, and Q can be expressed as the linear combination of other DMUs. DMU C is neither an extreme point of the efficiency surface, nor is it on the extended portion of the efficiency surface. Thus, $C \in E'$. We can see that DMU C can be expressed as the linear combination of DMUs B and D.

In Andersen and Petersen (1993), it is shown that the MDEA method was able to differentiate the efficient DMUs in $E' \cup F$ and those in E. The DMUs in E are assigned efficiency scores strictly greater than (less than) one in the input-oriented (output-oriented) MDEA models. Meanwhile, the DMUs in $E' \cup F$ are assigned the same efficiency scores of one in the MDEA models as in the conventional DEA models. It was also shown that MDEA could differentiate among the efficient DMUs in E in that they could be assigned different efficiency scores in the MDEA models.

In this example, under input orientation, DMUs A and C are assigned the same efficiency scores of one in the MDEA models since $A \in F$ and $C \in E'$. DMUs B and D are assigned different efficiency scores, which are both greater than one in the input-oriented MDEA model (less than one in the output-oriented model) since B and D $\in$ E. According to Andersen and Petersen (1993), since DMU Q$\in$E, DMU Q should also be assigned a new efficiency score greater than one in the input-oriented MDEA model. However, for the reasons we discussed before, the primal problem for DMU Q in the input-oriented MDEA model is infeasible. The reason is that DMU Q is actually a super efficient DMU.

3.2 Super Efficient DMUs in the Input-oriented MDEA VRS Model

Let us begin with the following definition:

Definition 1: Super Efficient in the Input-Oriented MDEA VRS Model

Consider the input-oriented Modified DEA (MDEA) variable returns-to-scale (VRS) primal model for DMU k given in Model 1. If $\forall\ (\theta_{k,} \vec{\lambda}, \vec{s}^{\,+}, \vec{s}^{\,-})$ which satisfies (2)-(9) and (11), there exists at least one component $\bar{n} \in N$ of the output vector, such that

$$\sum_{i \in I, i \neq k} \lambda_i Y_{i\bar{n}} < Y_{k\bar{n}} \quad \bar{n} \in N\ , \tag{34}$$

DMU k is defined as *super efficient in the input-oriented MDEA VRS model*.

The set of super efficient DMUs in the input-oriented MDEA VRS model is denoted by SEI.

Proposition 1

In the input-oriented MDEA VRS primal model for DMU k given in Model 1, the primal problem for DMU k is infeasible if and only if DMU k is super efficient in the input-oriented MDEA VRS model.

Proof of Proposition 1:

(Sufficiency) Suppose DMU k (k$\in$I) is super efficient in the input-oriented MDEA VRS model. By Definition 1, in the input-oriented MDEA VRS primal model for DMU k given in Model 1, $\forall(\theta_{k,} \vec{\lambda}, \vec{s}^{\,+}, \vec{s}^{\,-})$ which satisfies (2) –(9) and (11), there exists at least one component $\bar{n} \in N$ of the output vector such that $\sum_{i \in I, i \neq k} \lambda_i Y_{i\bar{n}} < Y_{k\bar{n}}$. Then, combining this result with (1) in Model 1, we have

$$s^{+}_{k\bar{n}} = \sum_{i\in I, i\neq k} Y_{in}\lambda_i - Y_{k\bar{n}} < 0 \quad \bar{n} \in N \,. \tag{35}$$

This result contradicts (10).

Hence, the primal problem for the super efficient DMU k in the input-oriented VRS MDEA model is infeasible.

(Necessity) Suppose that for DMU k ($k\in I$) the primal problem in the input-oriented MDEA VRS model is infeasible. Assume that DMU k is not input-oriented super efficient. Then, by Definition 1, $\forall\ (\theta_{k,}\vec{\lambda}, \vec{s}^{\,+}, \vec{s}^{\,-})$ which satisfies (2)-(9) and (11) in Model 1, there must exist

$$\sum_{i\in I, i\neq k} Y_{in}\lambda_i \geq Y_{kn} \ \ \forall\ n\in N \,. \tag{36}$$

Otherwise, DMU k is super input-oriented efficient. Combine (36) with constraint (1), we have

$$s^{+}_{kn} = \sum_{i\in I, i\neq k} Y_{in}\lambda_i - Y_{kn} \geq 0 \ \ \forall n\in N \,. \tag{37}$$

We can see that (37) is exactly the same as (10) in Model 1. Thus, all the constraints in Model 1 are satisfied. Therefore, the primal problem for DMU k is feasible. This is in contradiction with the assumption that the primal problem for DMU k is infeasible and hence, a contradiction is obtained.

Therefore, if the primal problem for DMU k given in Model 1 is infeasible, DMU k is super efficient in the input-oriented VRS MDEA model. □

Proposition 2

In the input-oriented modified DEA (MDEA) variable returns-to-scale dual model for DMU k given in Model 2, if the dual problem for DMU k is unbounded, DMU k is super efficient in the input-oriented MDEA VRS model.

Proof of Proposition 2:

According to standard results in duality theory, if the dual problem is unbounded, the primal problem is infeasible (see, for example, Theorem 2.7.3(b), Bazaraa, Sherali and Shetty 1993). Combining this with Proposition 1, we can conclude that if the dual problem for DMU k given in Model 2 is unbounded, DMU k is super efficient in the input-oriented MDEA VRS model. □

Proposition 3

Consider the input-oriented modified DEA (MDEA) variable returns-to-scale models for DMU k given in Model 1 and Model 2. If DMU k is super efficient in the input-oriented MDEA VRS model, it is strongly efficient. That is, under input-oriented model orientation, SEI$\subset$ E.

Proof of Proposition 3:

Consider the input-oriented MDEA VRS model. If DMU k is super efficient, by Definition 1 and Proposition 1, no convex combination of any other DMUs in the reference set can produce at least the same output as DMU k does by using at most the same input. Hence, according to the definition of strongly efficient DMU, DMU k is strongly efficient. Therefore, we have SEI$\subset$ E in the input-oriented MDEA VRS model. $\square$

Definition 2: Strongly Super Efficient in the Input-oriented MDEA VRS Model

If DMU k strictly exceeds any other DMUs in at least one dimension of the output vector, that is, there exists at least one component $\bar{n} \in N$ of the output vector such that

$$Y_{i\bar{n}} < Y_{k\bar{n}} \quad \forall i \in I, i \neq k, \tag{38}$$

DMU k is defined as *strongly super efficient in the input-oriented MDEA VRS model*.

The set of the strongly super efficient DMUs in the input-oriented MDEA VRS model is denoted by SSEI.

Proposition 4

If DMU k is a strongly super efficient DMU in the input-oriented MDEA VRS model, it is super efficient in the input-oriented MDEA VRS model. That is, SSEI $\subset$ SEI.

Proof of Proposition 4:

Assume that DMU k (k$\in$I) is strongly super efficient in the input-oriented MDEA VRS model. By Definition 2, there exists at least one component $\bar{n} \in N$ of the output vector such that $Y_{i\bar{n}} < Y_{k\bar{n}} \quad \forall\, i \in I, i \neq k$. Also, by (5), (6) and (7) we know that

$$0 \le \lambda_i \le 1 \;\; \forall i \in I. \tag{39}$$

Thus, we have

$$\lambda_i Y_{k\bar{n}} > \lambda_i Y_{i\bar{n}} \ \forall i \in I, i \neq k . \quad (40)$$

Consequently, we have

$$\sum_{i \in I} \lambda_i Y_{k\bar{n}} > \sum_{i \in I, i \neq k} \lambda_i Y_{i\bar{n}} . \quad (41)$$

Then by (5) and (41) we have $\sum_{i \in I, i \neq k} \lambda_i Y_{i\bar{n}} < Y_{k\bar{n}}$. Thus by Definition 1, DMU k is super efficient in the input-oriented MDEA VRS model. Therefore, a strongly super efficient DMU is always super efficient in the input-oriented MDEA VRS model. Hence, SSEI ⊂ SEI. □

<u>Proposition 5</u>

If DMU k is strongly super efficient in the input-oriented MDEA VRS model, the primal model for DMU k given in Model 1 is infeasible.

<u>Proof of Proposition 5:</u>

Combining Definition 1, Definition 2, Proposition 1 and Proposition 4, it is obvious that as long as DMU k is strongly super efficient in the input-oriented MDEA VRS model, the primal problem for DMU k given in Model 1 is infeasible. □

Remark: By using Definition 2 and Proposition 5, once we identify that DMU k is strongly super efficient in the input-oriented MDEA VRS model, we can conclude that its primal problem is infeasible and we do not need actually run the primal model.

The definitions and propositions about super efficient DMUs in the output-oriented MDEA VRS model can be defined in an analogous manner.[3]

Generally, if we denote the set of the super efficient DMUs by SE and the set of the strongly super efficient DMUs by SSE, then SE = SEI∪SEO, SSE = SSEI∪SSEO and SSE⊂SE⊂E. Considering the MDEA under certain model orientation, the relative efficiency of the efficient DMUs in different subsets can be ranked from the highest to the lowest as: Super Efficient → Strongly Efficient → Efficient → Weakly Efficient (SE→ E→E'→F).

[3] The statements of these propositions can be found on-line at the following address: http://opim.wharton.upenn.edu/~harker/MDEAAppendix.pdf

4. Efficiency Score Relationship

In Proposition 1 in Banker and Morey (1986), it is stated that the efficiency score for a observation from the input-oriented BCC model with certain inputs as non-discretionary is less than or equal to the efficiency score from the corresponding input-oriented BCC model with those inputs as discretionary. However, we cannot generally extend this property to the MDEA case because in MDEA, the efficiency scores are not necessarily less than or equal to one. For the inefficient DMUs, this proposition still holds. However, in the case of the efficient DMUs, things become more complicated.

In this section, we will investigate the general relationship between the efficiency scores from the input-oriented MDEA VRS model with certain inputs as non-discretionary (Model 1) and the efficiency scores from the corresponding input-oriented MDEA VRS model with those inputs as discretionary (Model 3).

First, we will investigate whether the DMUs will change their efficiency classes after some of the inputs are treated as non-discretionary.

Proposition 6

Assume that ρ_k is the efficiency score for DMU k from Model 1 and θ_k is the efficiency score from the corresponding Model 3 and that $X_{im} > 0 \; \forall \, i \in I, m \in M$. Then,

I. If $\theta_k = 1$, then $\rho_k \le 1$; and if $\rho_k = 1$, then $\theta_k = 1$.

II. If $\theta_k < 1$, then $\rho_k < 1$; and if $\rho_k < 1$, then $\theta_k \le 1$.

III. $\rho_k > 1$, if and only if $\theta_k > 1$.

IV. If $\theta_k = +\infty$, then $\rho_k = +\infty$; and if $\rho_k = +\infty$, then $1 < \theta_k \le +\infty$.

Proof of Proposition 6:

Assume that the optimal solution to Model 3 is $(\theta_{k,} \vec{\lambda}^T, \vec{s}^{\,T+}, \vec{s}^{\,T-})$ and the optimal solution for Model 1 is $(\rho_{k,} \vec{\lambda}^R, \vec{s}^{\,R+}, \vec{s}^{\,R-})$. Also assume that DMU T is the projection of DMU k onto the projection surface (including the extended portion of the efficiency surface) in Model 3 while DMU

R is the projection of DMU k onto the projection surface (including the extended portion of the efficiency surface) in Model 1. By assuming DMU T and DMU R as DMU k's projections in Model 3 and Model 1 respectively, we mean:

$$\sum_{i\in I, i\neq k} \lambda_i^T Y_{in} - s_{kn}^{T+} = Y_{Tn} \quad \forall\, n\in N \tag{42}$$

$$\sum_{i\in I, i\neq k} \lambda_i^T X_{id} + s_{kd}^{T-} = X_{Td} \quad \forall\, d\in D \tag{43}$$

$$\sum_{i\in I, i\neq k} \lambda_i^T X_{if} + s_{kf}^{T-} = X_{Tf} \quad \forall\, f\in F \tag{44}$$

$$\sum_{i\in I, i\neq k} \lambda_i^R Y_{in} - s_{kn}^{R+} = Y_{Rn} \quad \forall\, n\in N \tag{45}$$

$$\sum_{i\in I, i\neq k} \lambda_i^R X_{id} + s_{kd}^{R-} = X_{Rd} \quad \forall\, d\in D \tag{46}$$

$$\sum_{i\in I, i\neq k} \lambda_i^R X_{if} + s_{kf}^{R-} = X_{Rf} \quad \forall\, f\in F\,. \tag{47}$$

Then from Model 3 and Model 1, we have

$$\theta_k X_{kf} = X_{Tf} \quad \forall\, f\in F \tag{48}$$

$$\theta_k X_{kd} = X_{Td} \quad \forall\, d\in D \tag{49}$$

$$Y_{kn} = Y_{Tn} \quad \forall\, n\in N \tag{50}$$

$$X_{kf} = X_{Rf} \quad \forall\, f\in F \tag{51}$$

$$\rho_k\, X_{kd} = X_{Rd} \quad \forall\, d\in D \tag{52}$$

$$Y_{kn} = Y_{Rn} \quad \forall\, n\in N\,. \tag{53}$$

With the assumption that $X_{im} > 0\ \forall\, i\in I, m\in M$, we have $X_{Rf} > 0$. Then by (48) and (51), we have

$$\theta_k = X_{Tf} / X_{Rf} \quad \forall\, f\in F\,. \tag{54}$$

Also with the assumption that $X_{im} > 0\ \forall\, i\in I, m\in M$, by (48) and (52) we have $\theta_k > 0$ and $\rho_k > 0$. By (49) and (52), we have

$$\theta_k / \rho_k = X_{Td} / X_{Rd} \quad \forall\, d\in D\,. \tag{55}$$

By (50) and (53), we have

$$Y_{Tn} = Y_{Rn} \quad \forall\, n \in N \,. \tag{56}$$

Obviously, as DMU k's projections on the efficiency surface, both DMU T and DMU R are efficient.

I. First, we show that if $\theta_k = 1$, then $\rho_k \le 1$.

Given $\theta_k = 1$, by comparing the constraints in Model 3 and Model 1, we can see that $(\theta_k, \vec{\lambda}^T, \vec{s}^{T+}, \vec{s}^{T-})$ is a feasible solution to Model 1. Since the objective of Model 3 is to minimize the efficiency score, the optimal efficiency score ρ_k for DMU k in Model 1 should be less than or equal to one. Thus we proved that if $\theta_k = 1$, then $\rho_k \le 1$.

Second, we show that if $\rho_k = 1$, then $\theta_k = 1$.

Given $\rho_k = 1$, assume that $\theta_k < 1$. By (54) and (55), we have

$$X_{Tf} < X_{Rf} \quad \forall\, f \in F \tag{57}$$

$$X_{Td} < X_{Rd} \quad \forall\, d \in D \,. \tag{58}$$

By (56), (57), and (58) we can conclude that DMU R is inefficient. Then, a contradiction is obtained. Similarly, we can prove that if $\theta_k > 1$, DMU T is inefficient and thus another contradiction is obtained. We have thus established that if $\rho_k = 1$, then $\theta_k = 1$.

Therefore, we have proved Proposition 6.I.

II. First we show that if $\theta_k < 1$, then $\rho_k < 1$.

Given $\theta_k < 1$, assume that $\rho_k \ge 1$. By (54) and (55) we have

$$X_{Tf} < X_{Rf} \quad \forall\, f \in F \tag{59}$$

$$X_{Td} < X_{Rd} \quad \forall\, d \in D \,. \tag{60}$$

By (56), (59), and (60), we can conclude that DMU R is inefficient. Then, a contradiction is obtained. Thus, we proved that if $\theta_k < 1$, then $\rho_k < 1$.

Second we show that if $\rho_k < 1$, then $\theta_k <= 1$.

Given $\rho_k < 1$, assume that $\theta_k > 1$. By (54) and (55) we have

$$X_{Tf} > X_{Rf} \quad \forall\ f \in F \tag{61}$$

$$X_{Td} > X_{Rd} \quad \forall\ d \in D. \tag{62}$$

By (56), (61), and (62), we can conclude that DMU T is inefficient. Then a contradiction is obtained. Thus, we proved that if $\rho_k < 1$, then $\theta_k \leq 1$.

Therefore, we have proved Proposition 6.II.

III. (Sufficiency) Given $\theta_k > 1$, assume that $\rho_k \leq 1$. By (54) and (55), we have

$$X_{Tf} > X_{Rf} \quad \forall\ f \in F \tag{63}$$

$$X_{Td} > X_{Rd} \quad \forall\ d \in D. \tag{64}$$

By (56), (63), and (64) we can conclude that DMU T is inefficient. Then, a contradiction is obtained. Thus, we have proved that if $\theta_k > 1$, then $\rho_k > 1$.

(Necessity) First, given $\rho_k > 1$, it is obvious that $\theta_k \neq 1$, or else, by Proposition 6.I (proved above), we have $\rho_k \leq 1$, which contradicts the given condition $\rho_k > 1$.

Second, given $\rho_k > 1$, assume that $\theta_k < 1$. By (54) and (55), we have

$$X_{Tf} < X_{Rf} \quad \forall\ f \in F \tag{65}$$

$$X_{Td} < X_{Rd} \quad \forall\ d \in D. \tag{66}$$

By (56), (65) and (66), we can conclude that DMU R is inefficient. Then a contradiction is obtained. Thus, we proved that if $\rho_k > 1$, then $\theta_k > 1$.

Therefore, we have proved Proposition 6.III.

IV. First, we show that if $\theta_k = +\infty$, then $\rho_k = +\infty$.

Given $\theta_k = +\infty$, using the results in Section 2, we know that for DMU k, the primal problem in Model 3 is infeasible. Compare the constraints in Model 3 and Model 1, we can see that Model 1 is more restricted than Model 3. Thus, the primal problem for DMU k given in Model 1 should be infeasible as well. By Proposition 1 in Section 3, we know that DMU k should be evaluated as super efficient in the input-oriented Model 1 as well.

Therefore, the efficiency score for DMU k in Model 1 is also positive infinity. Hence, we have proved that if $\theta_k = +\infty$, then $\rho_k = +\infty$.

Second, we show that if $\rho_k = +\infty$, then $1 < \theta_k \leq +\infty$.

Given that $\rho_k = +\infty > 1$, by Proposition 6.III (proved above), we have that $\theta_k > 1$, which is equal to $1 < \theta_k \leq +\infty$. Thus we have proved that if $\rho_k = +\infty$, then $1 < \theta_k \leq +\infty$.

Therefore, we have proved Proposition 6.IV. ◻

Remarks:

1. For the conclusions and proof of Proposition 6 to hold, it is necessary to assume that $X_{im} > 0 \ \forall\, i \in I, m \in M$, that is, there is no zero in the input data. The extension of the methodology presented in this paper to the special case that there are some zeros in the input data is left for future research.
2. Given that the optimal solution for Model 3 is $(\theta_k, \vec{\lambda}^T, \vec{s}^{T+}, \vec{s}^{T-})$, only when $\theta_k = 1$ will this solution also be a feasible solution for Model 1. When $\theta_k < 1$ or $\theta_k > 1$, this optimal solution $(\theta_k, \vec{\lambda}^T, \vec{s}^{T+}, \vec{s}^{T-})$ for Model 3 is not a feasible solution for Model 1 because constraint (4) cannot hold:

   By (21), we have

   $$\theta_k X_{kf} - \sum_{i \in I, i \neq k} \lambda_i X_{if} - s_{kf}^- = 0 \quad \forall\ f \in F \subseteq M .$$

   (67)

   With the assumption that $X_{im} > 0 \ \forall\, i \in I, m \in M$, we have that $\theta_k > 0$. Then, by (67) we have

   $$X_{kf} = \frac{1}{\theta_k} \left(\sum_{i \in I, i \neq k} \lambda_i X_{if} + s_{kf}^- \right) \quad \forall\ f \in F \subseteq M . \qquad (68)$$

   Compare (68) and (4), we can see (4) holds only when $\theta_k = 1$.

<u>Proposition 7</u>

Assume ρ_k is the efficiency score for DMU k from Model 1 and θ_k is the efficiency score from the corresponding Model 3 and that $X_{im} > 0 \ \forall\, i \in I, m \in M$. Then,

I. If $\theta_k = 1$, then $\rho_k \le \theta_k$; and if $\rho_k = 1$, then $\theta_k = \rho_k$.

II. If $\theta_k < 1$, or $\rho_k < 1$, then $\rho_k \le \theta_k$; and if $\rho_k < \theta_k$, then $\rho_k < \theta_k < 1$.

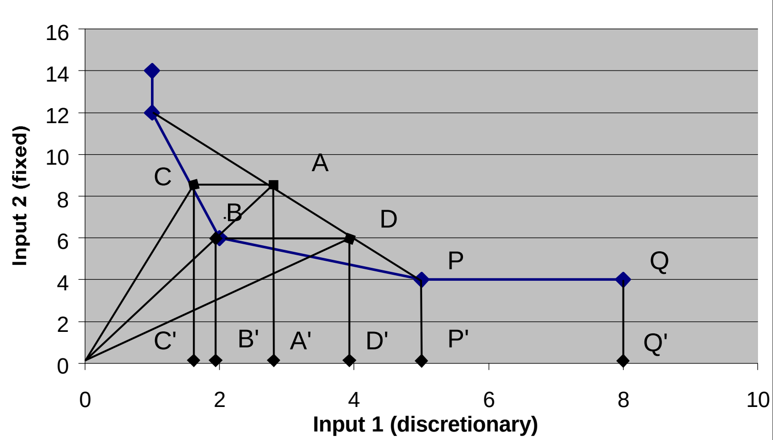
III. If $\theta_k > 1$, or $\rho_k > 1$, then $\rho_k \ge \theta_k$; and if $\rho_k > \theta_k$, then $\rho_k > \theta_k > 1$.

Proof for Proposition 7:

Using the conclusions of Proposition 6 and similar procedure to the proof for Proposition 6, Proposition 7 can be easily proved and hence, the details are omitted. ◻

To illustrate the results of Proposition 6 and 7, consider the example given in Figure 3. This example has one output and two inputs: Input 1 and Input 2. Input 1 is treated as discretionary in both Model 1 and Model 3, while Input 2 is treated as discretionary in Model 3 but non-discretionary in Model 1. All the DMUs in the reference set produce the same level output.

Figure 3. Example of Efficiency Score Relationship



From this example, one can see:

I. DMU C is efficient both in Model 3 and in Model 1 (C ∈ E'); that is, $\theta_c = \rho_c = 1$.

II. DMU A is inefficient both in Model 3 and in Model 1. This DMU projects to DMU B in Model 3; however, it projects to DMU C in Model 1. As $\theta_A = \frac{OB'}{OA'}$, $\rho_A = \frac{OC'}{OA'}$, and OC'< OB', we have $\theta_A > \rho_A$.

III. DMU B is strongly efficient both in Model 3 and in Model 1(B$\in$E). This DMU projects to DMU A in Model 3; however, it projects on DMU D in Model 1. As $\theta_B = \frac{OA'}{OB'}$, $\rho_B = \frac{OD'}{OB'}$, and OA'< OD', we have $\theta_B < \rho_B$.

IV. DMU Q is weakly efficient in Model 3 (Q $\in$ F), and it becomes inefficient in Model 1. In Model 1, DMU Q projects to DMU P, because its Input 2 is fixed its current level. As $\theta_Q = \frac{OQ'}{OQ'} = 1$, $\rho_Q = \frac{OP'}{OQ'}$, and OP'<OQ', we have $\rho_Q < \theta_Q = 1$.

Since the proofs of Proposition 6 and 7 are independent to the convexity constraint (5) in Model 1 and (22) in Model 3, the results in Proposition 6 and 7 are also applicable to the input-oriented constant returns-to-scale (CRS) MDEA models.

5. Empirical Illustration

In order to illustrate the usefulness of the results in the previous sections, the models proposed herein were applied to a set of performance and demographic data from fifty-two branches of a large commercial bank in the United States (see Appendix A for a description of the data set).

Table 3 lists the inputs and outputs used in this analysis. On the input side, the financial specialists are the bank employees in the branches who are primarily responsible for the sales of financial products. The customer base is measured by the total annual personal income of the county where the branch is located in 1993. On the output side, "financial revenue points" is a measure of aggregate revenue generated by the branch. The financial products included in this revenue calculation are investment products, loan products, Certificates of Deposits (CDs), and small business loans. This set of inputs and outputs is similar to the one used by Athanassopoulos (1998) in his analysis of branch efficiency

Table 3: Inputs and Output for Bank Example

Decision Making Units	52 branches of a large U.S. bank
Input 1: Financial Specialists	The number of Financial Specialists in the branch by the end of December 1997
Input 2: Customer Base	The total annual personal income of the county where the branch is located according to the census in 1993
Output 1: Financial Revenue	The financial revenue points of the branch in 1997

Using this data set, let us now illustrate the value of the methodology proposed in this paper.

5.1 Phase I: Illustration of the Properties of the Super Efficient DMUs

First, let us consider an example with a single discretionary input and a single output. We applied both the input-oriented BCC primal and dual models, and the input-oriented MDEA VRS primal and dual models (Model 3 and Model 4) to this example[4]. The ranks and efficiency scores from the two types of models are compared in Appendix B.

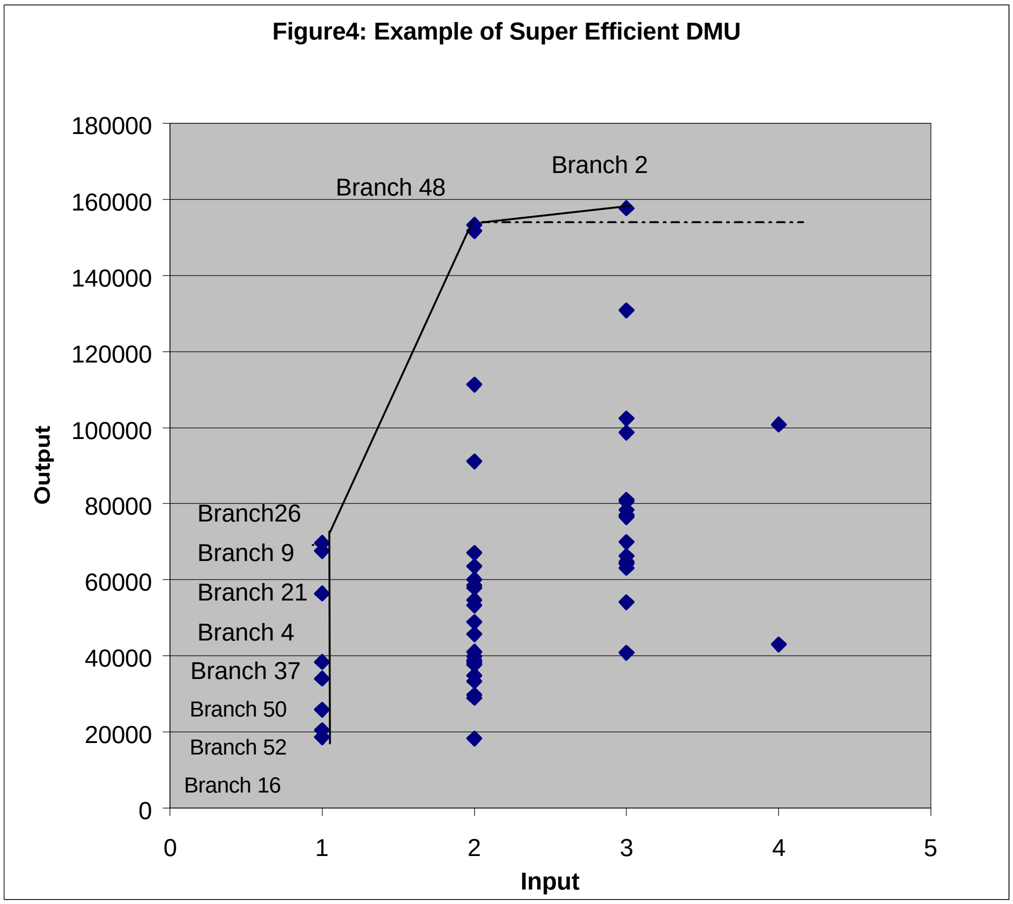
Table 4: Model Summary for Example 1

Model Type	Input-oriented BCC Model	Input-oriented VRS MDEA Model
Decision Making Units	52 branches	52 branches
Input	Input 1: Financial Specialists (discretionary)	Input 1: Financial Specialists (discretionary)
Output	Output 1: Financial Revenue	Output 1: Financial Revenue
Model Orientation	Input-oriented	Input-oriented

[4] The GAMS files as well as the input files for the models in this section can be downloaded from the following Internet site: http://opim.wharton.upenn.edu/~harker/MDEAGAMS.html. The solver we used is MINOS5.

Returns-to-scale	Variable Returns-to-scale	Variable Returns-to-scale

In Figure 4, the original efficiency frontier from the BCC model consists of Branches 2, 4, 9, 16, 21, 26, 37, 48, 50 and 52. As the vertices of the efficiency frontier, Branches 2, 26, 48 $\in$ E. Branches 4, 9, 16, 21, 37, 50 and 52 are weakly efficient in the input-oriented model but not efficient in the output-oriented model since Branch 26 uses the same input as these DMUs but produces more output than all of them. Therefore, Branches 4, 9, 16, 21, 37, 50 and 52 $\in$ F.



After applying the MDEA method, Branches 26 and 48 are assigned different efficiency scores greater than one, while Branches 4, 9, 16, 21, 37, 50 and 52 are all assigned the same efficiency scores equal to one (see Table B2). According to the MDEA methodology, since Branch 2 is also a vertex of the efficiency frontier (see Figure 4), it should be assigned a new efficiency score strictly greater than one in the input-oriented MDEA VRS model. However, the solution report from GAMS shows that in the input-oriented MDEA VRS model, the primal problem for Branch 2 is infeasible.

Here, the value of Branch 2's output, 154026, is strictly greater than any other branches' output. In an input-oriented model, if we exclude Branch 2 from the efficiency frontier, we can never project Branch 2 back onto the projection frontier (new efficiency frontier) along the direction of increasing input. Here the projection frontier consists of the segments connecting Branches 16, 52, 50, 37, 4, 21, 9, 26 and 48, as well as the extension parallel to the input axis, which is indicated by the dashed line. The reason is that no convex combination of the reduced reference set of DMUs can use the same or even more input as Branch 2's to produce the same output as Branch 2 does. From another perspective, we can say that Branch 2 can proportionately increase its input to positive infinity while remaining efficient. This leads Branch 2's new efficiency score to go to positive infinity. Since the objective of the primal model is to minimize Branch 2's efficiency score, the primal model for Branch 2 turns out to be infeasible.

Therefore, with a closer look, Branch 2 should be evaluated to be relatively more efficient than the other efficient branches under input-oriented model orientation. Since Branch 2's output is strictly greater than any other branch's output, by Definition 2, Branch 2 is not only super efficient but also strongly super efficient.

5.2 Phase II: Illustration of the Efficiency Score Relationship

Now, we add Input 2, the customer base measured by the total annual personal income in the county where the branch is located, to the model. Obviously, this input is non-discretionary at a branch manager's level. We apply four types of models as follows (Table 5):

In Types 1 and 2, both Input 1 and Input 2 are treated as discretionary inputs. In Types 3 and 4, however, Input 2 is treated as non-discretionary while Input 1 is treated as discretionary. In Appendix C, we compare the efficiency scores and ranks from these models.

Concerning the efficiency scores and the ranks, there are two properties we want to investigate:

(1) In Andersen and Petersen (1993), it is pointed out that after applying the MDEA method, the inefficient observations are assigned the same index (efficiency scores) as in the conventional constant returns-to-scale model (CCR Model).

In our example, all the inefficient branches have the same efficiency scores in Type 2 as in Type 1. This shows that this property can be extended to the MDEA model under variable returns-to-scale conditions. In addition, we also noted that all the inefficient branches had the same efficiency scores in Type 3 as in Type 4. This shows that this property is still true when non-discretionary inputs are considered.

Table 5: Model Summary of Example 2

Model Name		Type 1	Type 2	Type 3	Type 4
Model Type		Input–oriented BCC	Input-oriented VRS MDEA	Input-oriented BCC	Input-oriented VRS MDEA
Decision Making Units		the 52 branches	the 52 branches	the 52 branches	the 52 branches
Input	**Discretionary Input**	Input 1: Financial Specialists	Input 1: Financial Specialists	Input 1: Financial Specialists	Input 1: Financial Specialists
		Input 2: Customer Base	Input 2: Customer Base		
	Non-discretionary Input			Input 2: Customer Base	Input 2: Customer Base
Output		Output 1: Financial Revenue	Output 1: Financial Revenue	Output 1: Financial Revenue	Output 1: Financial Revenue
Model Orientation		Input-oriented	Input-oriented	Input-oriented	Input-oriented
Returns-to-scale		VRS	VRS	VRS	VRS

(2) The results also verified our Proposition 6 and 7 in Section 4. As we mentioned before, in Proposition 1 in Banker and Morey (1986), it is stated that after taking some input as non-discretionary, every DMU is assigned an efficiency score less than or equal to its score in the corresponding input-oriented BCC model. This property is verified here with the efficiency scores from Type 1 and Type 3. However, we cannot generally extend this property to the MDEA case. By comparing the efficiency scores from Type 2 and Type 4, we find:

I. For all the inefficient branches, their efficiency scores in Type 4 are less than or equal to the corresponding efficiency scores in Type 2.

II. Branches 4, 9, 16, 21, 37, 50 and 52 are all weakly efficient in both Type 2 and Type 4. They all have the same efficiency scores equal to one in Type 4 as in Type 2.

III. Both Branch 26 and Branch 49 are strongly efficient both in Type 2 and in Type 4. They both have the higher efficiency scores in Type 4 compared to their scores in Type 3.

IV. Branch 10 has an efficiency score equal to 3.305 in Type 2; but its primal problem becomes infeasible in Type 4. Branch 48 has an efficiency score equal to 1.329 in Type 2 but its primal problem also becomes infeasible in Type 4. That is, Branch 10 and Branch 48 are evaluated as just strongly efficient when we treat the total annual personal income as discretionary input. But, it turns out that they are actually not only strongly efficient, but also super efficient after considering Input 2 as non-discretionary.

V. Branch 2 is evaluated as super efficient in Type 4 as well as in Type 2. The primal problems for Branch 2 are infeasible both in Type 2 and in Type 4. As we discussed before, with its strictly highest level output, Branch 2 is strongly super efficient. Therefore, as a strongly super efficient DMU, Branch 2 has efficiency scores going to positive infinity before and after taking Input 2 as a non-discretionary input.

In general, we notice that before and after taking the non-discretionary input into account, there are dramatic changes of the efficiency scores and, especially, ranks both for the efficient branches and the inefficient branches. This shows that it is important to develop MDEA models with non-discretionary input.

6. Summary

As pointed out by Charnes, Cooper, Lewin and Seiford: “ The primary benefit of this approach is the ability to make finer distinctions between efficient DMUs and to produce a logarithmic MDEA distribution of relative performance scores that are approximately normally distributed.” (Charnes *et al.* 1994). MDEA has the potential to overcome the analytic difficulties to the post-DEA regression analysis posed by the spiked distribution of the DEA scores.

However, when we extend MDEA under variable returns-to-scale conditions, the possibility for the primal problem for some efficient DMUs to be infeasible can undermine the application of MDEA under variable returns-to-scale conditions, because we cannot assign efficiency scores to these efficient DMUs. In this case, the ranking of the entire DMU set is absolutely impossible. Based on our above discussion and proof, we showed that even in this case, we were still able to provide a full ranking based on the relative efficiency of the entire DMU set without losing any desirable property of the MDEA efficiency scores. With further investigation, we identified a special subset of the set of the strongly efficient DMUs, the super efficient DMUs(SE). In addition, we also identified a special subset of the super efficient DMUs, the strongly super efficient DMUs (SSE). Generally, the relative efficiency of units in the four classes can be ranked from higher to lower as:

Super Efficient → Strongly Efficient → Efficient → Weakly Efficient (SE→ E→E'→F),

where SSE⊂SE⊂E.

With the extension of MDEA models with non-discretionary inputs, we provide an even stronger tool to evaluate the efficiency of DMUs in practice. However, we cannot generally extend the current theorem about the relationship between the efficiency scores from the model with some of the inputs as non-discretionary and the ones from the corresponding model with these inputs as discretionary to the MDEA case. With Proposition 6 and Proposition 7, we provided by far the most complete picture about the relationship between the efficiency scores from the input-oriented MDEA model with some of the inputs as non-discretionary and the ones from the corresponding MDEA model with these inputs as discretionary.

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Appendix A: Input and Output Data

Table A1: Performance and Demographic Data of 52 Bank Branches in U.S.A.

Branch	Number of Financial Specialists	Total Annual Personal Income($1000)	Financial Revenue Point	Branch	Number of Financial Specialists	Total Annual Personal Income($1000)	Financial Revenue Point
1	3	20180650	69913	27	2	11832930	60123
2	3	20180650	154026	28	3	13069660	78325
3	2	11832930	38651	29	4	11832930	100813
4	1	13069660	38453	30	3	20180650	64795
5	2	13069660	39966	31	3	3323654	63153
6	2	20180650	151661	32	4	9219095	43023
7	2	20180650	111312	33	2	20180650	91134
8	3	11832930	81068	34	3	20180650	64244
9	1	9219095	67628	35	3	13069660	77101
10	3	1165995	76673	36	3	20180650	98709
11	3	11832930	102363	37	1	20180650	34066
12	2	13069660	58554	38	3	9219095	66252
13	3	20180650	76490	39	3	13069660	64323
14	2	11832930	28971	40	2	11832930	57928
15	2	20180650	67076	41	2	11832930	37772
16	1	20180650	18662	42	3	3323654	40900
17	2	9219095	45738	43	2	13069660	29751
18	3	20180650	80547	44	3	20180650	130838
19	2	9219095	33339	45	2	11832930	38885
20	3	11832930	54137	46	2	11832930	48958
21	1	11832930	56389	47	2	13069660	18339
22	2	9219095	41077	48	2	13069660	153276
23	2	9219095	53274	49	2	1910481	63588
24	2	9219095	34892	50	1	20180650	25894
25	2	9219095	38256	51	2	11832930	54738
26	1	9219095	69688	52	1	11832930	20491

Notes:

Number of Financial Specialists: the number of financial specialists in the branch by the end of December 1997

Total Annual Personal Income: the total annual personal income in the county where the branch is located in 1993 (Data Source: U.S. Census Bureau, 1996)

Financial Revenue Points: the financial revenue points earned by the branch by the end of December 1997(YTD)

Appendix B: Rank and Efficiency Scores for Example 1

Table B1: Input-oriented BCC Model

Rank	Branch	Efficiency Score	Rank	Branch	Efficiency Score
1	2	1	**15**	40	0.5
1	4	1	**15**	41	0.5
1	9	1	**15**	43	0.5
1	16	1	**15**	45	0.5
1	21	1	**15**	46	0.5
1	26	1	**15**	47	0.5
1	37	1	**15**	49	0.5
1	48	1	**15**	51	0.5
1	50	1	**35**	11	0.464
1	52	1	**36**	36	0.449
11	6	0.99	**37**	8	0.379
12	7	0.749	**38**	18	0.377
13	33	0.628	**39**	28	0.368
14	44	0.577	**40**	35	0.363
15	3	0.5	**41**	10	0.361
15	5	0.5	**42**	13	0.36
15	12	0.5	**43**	29	0.343
15	14	0.5	**44**	1	0.334
15	15	0.5	**45**	20	0.333
15	17	0.5	**45**	30	0.333
15	19	0.5	**45**	31	0.333
15	22	0.5	**45**	34	0.333
15	23	0.5	**45**	38	0.333
15	24	0.5	**45**	39	0.333
15	25	0.5	**45**	42	0.333
15	27	0.5	**52**	32	0.25

Table B2: Input-oriented MDEA VRS Model

Rank	Branch	Efficiency Score	Rank	Branch	Efficiency Score
1	2	*positive infinity	**9**	40	0.5
2	48	1.341	**9**	41	0.5
3	26	1.024	**9**	43	0.5
4	4	1	**9**	45	0.5
4	9	1	**9**	46	0.5
4	16	1	**9**	47	0.5
4	21	1	**9**	49	0.5
4	37	1	**9**	51	0.5
4	50	1	**35**	11	0.464
4	52	1	**36**	36	0.449
5	6	0.99	**37**	8	0.379
6	7	0.749	**38**	18	0.377
7	33	0.628	**39**	28	0.368
8	44	0.577	**40**	35	0.363
9	3	0.5	**41**	10	0.361
9	5	0.5	**42**	13	0.36
9	12	0.5	**43**	29	0.343
9	14	0.5	**44**	1	0.334
9	15	0.5	**45**	20	0.333
9	17	0.5	**45**	30	0.333
9	19	0.5	**45**	31	0.333
9	22	0.5	**45**	34	0.333
9	23	0.5	**45**	38	0.333
9	24	0.5	**45**	39	0.333
9	25	0.5	**45**	42	0.333
9	27	0.5	**52**	32	0.25

***Note**: The primal problem for Branch 2 is infeasible.

Appendix C: Rank and Efficiency Scores for Example 2

Table C1: Input-oriented BCC model without Non-discretionary Input

Rank	Branch	Efficiency Score	Rank	Branch	Efficiency Score
1	2	1	**25**	27	0.625
1	4	1	**25**	40	0.625
1	9	1	**25**	41	0.625
1	10	1	**25**	45	0.625
1	16	1	**25**	46	0.625
1	21	1	**25**	51	0.625
1	26	1	**33**	5	0.597
1	37	1	**33**	12	0.597
1	48	1	**33**	43	0.597
1	49	1	**33**	47	0.597
1	50	1	**37**	44	0.587
1	52	1	**37**	8	0.546
13	6	0.99	**39**	38	0.531
14	7	0.749	**40**	29	0.529
15	17	0.693	**41**	28	0.515
15	19	0.693	**42**	35	0.51
15	22	0.693	**43**	15	0.5
15	23	0.693	**44**	20	0.49
15	24	0.693	**45**	36	0.487
15	25	0.693	**46**	39	0.472
21	31	0.655	**47**	18	0.43
21	42	0.655	**47**	32	0.43
23	11	0.63	**49**	13	0.417
24	33	0.628	**50**	1	0.397
25	3	0.625	**51**	30	0.393
25	14	0.625	**51**	34	0.393

Table C2: Input-oriented MDEA VRS Model without Non-discretionary Input

Rank	Branch	Efficiency Score	Rank	Branch	Efficiency Score
1	2	*positive infinity	**25**	27	0.625
2	10	3.305	**25**	40	0.625
3	48	1.53	**25**	41	0.625
4	49	1.329	**25**	45	0.625
5	26	1.024	**25**	46	0.625
6	4	1	**25**	51	0.625
6	9	1	**33**	5	0.597
6	16	1	**33**	12	0.597
6	21	1	**33**	43	0.597
6	37	1	**33**	47	0.597
6	50	1	**37**	44	0.587
6	52	1	**38**	8	0.546
13	6	0.99	**39**	38	0.531
14	7	0.749	**40**	29	0.529
15	17	0.693	**41**	28	0.515
15	19	0.693	**42**	35	0.51
15	22	0.693	**43**	15	0.5
15	23	0.693	**44**	20	0.49
15	24	0.693	**45**	36	0.487
15	25	0.693	**46**	39	0.472
21	31	0.655	**47**	18	0.43
21	42	0.655	**47**	32	0.43
23	11	0.63	**49**	13	0.417
24	33	0.628	**50**	1	0.397
25	3	0.625	**51**	30	0.393
25	14	0.625	**51**	34	0.393

***Note**: The primal problem for Branch 2 is infeasible.

Table C3: Input-oriented BCC Model with Non-discretionary Input

Rank	Branch	Efficiency Score	Rank	Branch	Efficiency Score
1	2	1	**18**	23	0.5
1	4	1	**18**	24	0.5
1	9	1	**18**	25	0.5
1	10	1	**18**	27	0.5
1	16	1	**18**	40	0.5
1	21	1	**18**	41	0.5
1	26	1	**18**	43	0.5
1	37	1	**18**	45	0.5
1	48	1	**18**	46	0.5
1	49	1	**18**	47	0.5
1	50	1	**18**	51	0.5
1	52	1	**38**	11	0.464
13	6	0.99	**39**	36	0.449
14	7	0.749	**40**	8	0.379
15	33	0.628	**41**	28	0.368
16	31	0.602	**42**	35	0.363
16	42	0.602	**43**	13	0.36
17	44	0.577	**44**	29	0.343
18	3	0.5	**45**	18	0.337
18	5	0.5	**46**	1	0.334
18	12	0.5	**47**	30	0.333
18	14	0.5	**47**	34	0.333
18	15	0.5	**47**	38	0.333
18	17	0.5	**47**	39	0.333
18	19	0.5	**51**	20	0.33
18	22	0.5	**52**	32	0.25

Table C4: Input-oriented MDEA VRS Model with Non-discretionary Input

Rank	Branch	Efficiency Score	Rank	Branch	Efficiency Score
1	2	*positive infinity	**19**	23	0.5
1	10	*positive infinity	**19**	24	0.5
1	48	*positive infinity	**19**	25	0.5
4	49	1.408	**19**	27	0.5
5	26	1.038	**19**	40	0.5
6	4	1	**19**	41	0.5
6	9	1	**19**	43	0.5
6	16	1	**19**	45	0.5
6	21	1	**19**	46	0.5
6	37	1	**19**	47	0.5
6	50	1	**19**	51	0.5
6	52	1	**38**	11	0.464
13	6	0.99	**39**	36	0.449
14	7	0.749	**40**	8	0.379
15	33	0.628	**41**	28	0.368
16	31	0.602	**42**	35	0.363
16	42	0.602	**43**	13	0.36
18	44	0.577	**44**	29	0.343
19	3	0.5	**45**	18	0.337
19	5	0.5	**46**	1	0.334
19	12	0.5	**47**	30	0.333
19	14	0.5	**47**	34	0.333
19	15	0.5	**47**	38	0.333
19	17	0.5	**47**	39	0.333
19	19	0.5	**51**	20	0.33
19	22	0.5	**52**	32	0.25

***Note**: The primal problems for Branches 2, Branch 10 and Branch 48 are infeasible.